SCARBOROUGH CAMPUS
UNIVERSITY OF TORONTO

MATA26Y April 23, 1996

FINAL EXAMINATION

1. Find the following antiderivatives: (Remember that you can check your answer!)

[4] (a) $\int \frac{4+x}{\sqrt{x}}\,dx$

[4] (b) $\int \frac{1+\sin(2x)}{2}\,dx$

[4] (c) $\int \frac{1}{(1+x)^2}\,dx$

[4] (d) $\int \frac{\ln|1+x|}{1+x}\,dx$

[4] (e) $\int \log_3\left(\frac{1}{x}\right)dx$

2. Find the derivatives of the following functions:

[2] (a) $f(x) = x^2\ln(x^2)$

[2] (b) $g(x) = \frac{x^2+4}{x^2-4}$

[2] (c) $h(x) = (x^2+1)\arctan x$

[2] (d) $m(x) = \sin\left(\cos(e^{3x})\right)$

[2] (e) $n(x) = \arcsin\left(2^{\sqrt{x}}\right)$

[10] 3. Let $S(t)$ be the number of daylight hours, in Cambridge, MA, at the t^{th} day of the year. During spring (from the vernal equinox, $t = 80$, to the summer solstice, $t = 173$), the graph of $S(t)$ is concave down — and of course increasing. In the table to the right we list some values of $S(t)$. What is the average length of the days in spring (in hours and minutes)? Please give an upper bound and a lower bound for the answer. These bounds should differ by less than 10 minutes. To begin with, sketch a possible graph of $S(t)$!

t	$S(t)$
80	12 hours 12 minutes
111	13 hours 44 minutes
142	14 hours 50 minutes
173	15 hours 12 minutes

4. The globular cluster M13 is a spherical distribution of stars which orbits our galaxy. Suppose that the density of stars in the cluster is purely a function of distance r from the center of the cluster and is given as

$$\rho(r) = \left(1 + \left(\frac{r}{100}\right)^3\right)^{-5} \frac{\text{stars}}{(ly)^3}$$

where r is measured in light-years, and $0 \le r \le 100ly$. (One light-year is the distance light travels in one year; "light-year" is abbreviated as "ly".)

[3] (a) How many stars will there be approximately in a spherical shell at radius r and of thickness Δr?

[2] (b) Set up an integral whose value is the number of stars in M13.

[5] (c) Evaluate the integral in part (b) to compute the total number of stars in M13. If you *insist* to evaluate the integral numerically (e.g. on your calculator), you will get credit for part (c) only if you:

(i) Give a number which is within 10,000 stars of the true value of the integral.

(ii) Give justification for your answer being accurate to within 10,000 stars.

[10] 5. Is $\displaystyle\int_{-1}^{\infty} \frac{e^{-x}\sin^2 x}{2+|x|}\,dx$ convergent or divergent? Give reasons for your answer and if it is convergent, give an upper bound for its value.

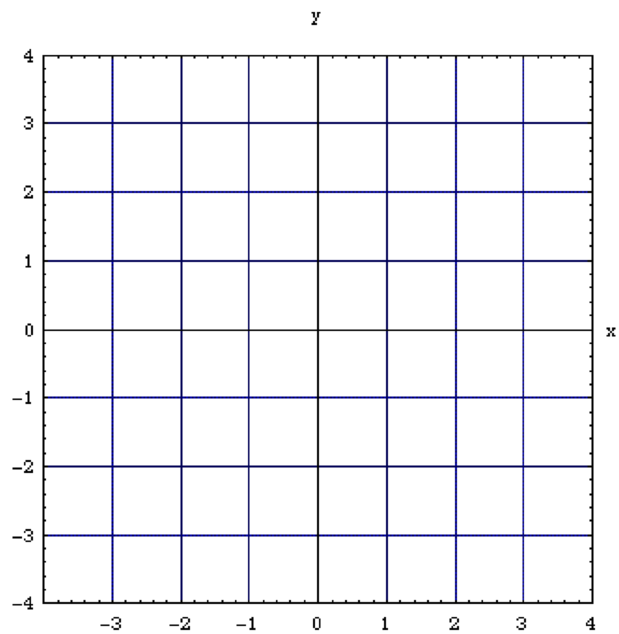
[10] 6. Find the *function* $y = f(x)$ that solves the differential equation $y' = \dfrac{-5}{1+y}$, and satisfies $f(0) = 2$. What is the domain of f?

7. This problem concerns the differential equation

$$\frac{dy}{dx} = -\frac{1}{3}(2x + y + 4) \ .$$

[5] (a) Show that $y = -2x+2$ is the unique solution of the equation that is a straight line. (Write $y = ax + b$ and show that a must equal -2 and b must equal 2.)

[5] (b) Give a reasonable sketch of the direction field of the equation. It helps to take account of your answer to part (a) and to consider what happens along the lines $y = -2x + 2 + c$, c a constant.



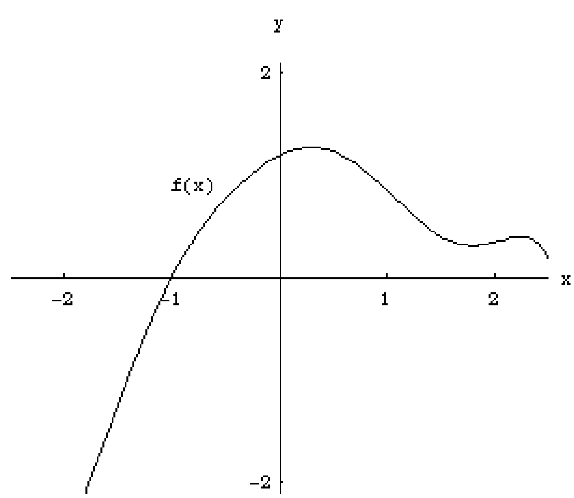
[5] (c) Show that

$$y = -2\left(x - 1 + Ce^{-\frac{x}{3}}\right)$$

is a solution for *any* constant C.

[5] (d) Find the solution passing through the point $(0,0)$ and describe its qualitative behavior as $x \to \pm\infty$.

8. Using the following graph



[6] (a) list in increasing order (from smallest to largest)

$$f'(2), \quad f(0), \quad f'(-0.9), \quad \text{the number } 1, \quad f''(0), \quad f''(1)\ .$$

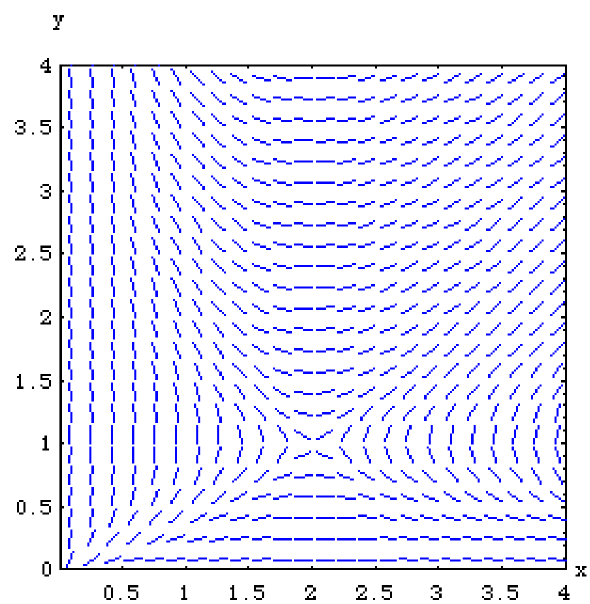
Don't forget to give reasons for your ordering!

[**4**] (b) Suppose we want to estimate $f(1.5)$ by using tangent line approximations at $x = 0, 1$ and 2. Which tangent line yields the best approximation?

9. Suppose the equations $\frac{dy}{dt} = 2y - xy$ and $\frac{dx}{dt} = x - xy$ describe the rates of growth of two interacting species.

[**3**] (a) In one sentence, summarize the nature of the interaction between these two species.

[**8**] (b) The slope field in the xy–phase plane is shown below. Sketch the trajectory for the initial conditions $x = 1$, $y = 2$. Be sure to indicate with arrows which direction along the trajectory the populations go in time, and show how you found this direction.



[**4**] (c) Describe the long-run behavior of the two populations with the initial conditions given in part (b).

[**5**] (d) Indicate on the slope field the nullclines and equilibrium points. Is there a stable equilibrium?