University of Toronto at Scarborough
Physical Sciences Division, Mathematics

MATA26Y | April 18, 2000
3 hours

FINAL EXAMINATION

[10] 1. Sketch the graph of the function $f(x) = x(x+2)^{2/3}$ showing extrema, points of inflection, intervals of increase and decrease and intervals of concavity. Show your work.

2. Calculate the derivatives of

[4] (a) $g(x)$, at $x = 2$, where $g(x)$ is the inverse function of $f(x) = e^x - e^{-x}$.
[*Note:* $f(0) = 2$]

[2] (b) $f(x) = \sin\left(\cos(\tan x^2)\right) \qquad x \in \left(-\frac{\sqrt{\pi}}{\sqrt{2}}, \frac{\sqrt{\pi}}{\sqrt{2}}\right)$

[4] (c) $f(x) = \displaystyle\int_{\tan x}^{x^2} \frac{1}{\sqrt{2+t^4}}\, dt.$

3. Find the following limits:

[2] (a) $\displaystyle\lim_{x\to\pi^-} \frac{|\pi - x|}{x - \pi}$

[4] (b) $\displaystyle\lim_{x\to\infty} \frac{x^{3/2} + 2x^2 - 4}{x \ln x}$

[4] (c) $\displaystyle\lim_{x\to 0} \frac{\tan x - \sin x}{x^3 \tan x}$

[6] 4. Find exactly how many roots $f(x) = 3x^5 - 5x^3 + 1$ has. *[Do not attempt to calculate the roots.]*

[4] 5. (a) Give the n^{th} degree Taylor polynomial of the function $f(x)$ at a and the n^{th} degree remainder term $R_n(x)$ associated with it.

[3] (b) Find the Taylor series for $f(x) = e^{-x}$ where $a = 0$.

[3] (c) Approximate $e^{-0.4}$ correct to four decimal places using your results from part (b).

6. Evaluate (or show the divergence of) each of the following integrals

[2] (a) $\displaystyle\int \frac{1}{2z^2 + 4z + 11}\, dz$

[2] (b) $\displaystyle\int \sin^3\theta\, d\theta$

[2] (c) $\displaystyle\int \frac{\ln x}{x}\, dx$

[3] (d) $\displaystyle\int \frac{x^4}{x^2 + 4x + 4}\, dx$

[4] (e) $\displaystyle\int_0^1 \frac{dx}{\sqrt{x(1-x)}}$

[4] 7. (a) State Simpson's Rule and give the error estimate.

[4] (b) Use Simpson's rule with $n = 4$ to approximate $\int_0^{\pi/4} \ln(\cos x)dx$.

[4] (c) Use the error formula for Simpson's rule to estimate the error for the approximation you found in (b).

[4] 8. (a) Find the length of the curve $y = \frac{1}{6}(x^3 + 3\,x^{-1})$ over the interval $1 \leq x \leq 3$.

[5] (b) Find the volume of the solid obtained by revolving about the y-axis the region bounded by the x-axis, the lines $x = 2\pi$, $x = 6\pi$, and the graph of $f(x) = 1 + \sin x$. [Hint: use the shell method.]

[2] 9. (a) Define: *monotone sequence.*

[1] (b) Define: *bounded sequence.*

[1] (c) State the *bounded monotone sequence theorem.*

(d) Determine whether or not each of the following sequences converges or diverges. If it converges, determine the limit.

[2] (i) $a_n = f(n)$ for $n \geq 0$, where $f(x) = \frac{1-x^2}{2+3x^2}$.

[4] (ii) $a_1 = 1$ and $a_{n+1} = \sqrt{1 + a_n}$ for $n \geq 1$.

[2] 10. (a) Define: *alternating series.*

[2] (b) State the theorem giving conditions under which an alternating series converges.

11. For each of the following series, determine whether it converges or diverges, and give your reasons.

[2] (a) $\sum_{n=1}^{\infty} n^{1/n}$

[2] (b) $\sum_{n=1}^{\infty} ne^{-n}$

[2] (c) $\sum_{n=1}^{\infty} (-1)^n \frac{1}{\sqrt{n}}$