University of Toronto at Scarborough
Physical Sciences Division, Mathematics

FINAL EXAMINATION
MATA26Y
Calculus

Examiners: X. Jiang R. Sharpe S. Tryphonas

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Duration: 3 hours

1. (a) **[3 pts]** State the Intermediate Value Theorem (IVT)

   (b) **[2 pts]** Does IVT apply to the function

   $$f(x) = \frac{x^2 - 3}{x^2 + x - 2}$$

   on the interval $[0, 2]$?

   (c) **[3 pts]** Show that the equation $x^5 + 2x - 2 = 0$ has exactly one root on the interval $[0, 1]$.

   (d) **[3 pts]** Show that $x^5 - x^3 + 4x^2 - 4 = 0$ has all its roots in the interval $[-9, 9]$.

2. Consider the curve given by $x^{\frac{2}{3}} + y^{\frac{2}{3}} = 5$.

   (a) **[1 pt]** Verify that the point $P_0 = (8, 1)$ lies on the curve.

   (b) **[3 pts]** Calculate the tangent line to the curve at P_0. (Put your answer in the form $y = mx + b$.)

3. **[15 pts]** Sketch the graph of $f(x) = \dfrac{x^2}{x^2 + e^x}$, being sure to show:

   (a) The regions where the function is positive and negative, increasing and decreasing. (Do not attempt to determine the concavity.)

   (b) The places where the function is zero and the critical points. (Do not attempt to find the inflection points.)

   (c) Local and global maxima and minima (if any).

   (d) Horizontal asymptotes (if any).

4. Calculate the following integrals

   (a) **[3 pts]** $\displaystyle\int \frac{e^{\sqrt{x+4}}}{\sqrt{x+4}}\,dx$ (b) **[7 pts]** $\displaystyle\int \frac{x^4}{x^2+4x+4}\,dx$

5. Evaluate each of the following integrals or show divergence.

   (a) **[2 pts]** $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{1+x^2}$ (b) **[4 pts]** $\displaystyle\int_0^2 \frac{dx}{\sqrt{2x-x^2}}$

6. **[8 pts]** When the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ is rotated about the x axis it generates an ellipsoid. Compute the volume of the ellipsoid.

7. **[8 pts]** Find the Taylor polynomial of degree 2 at $a=16$ for $f(x)=\sqrt{x}$ and use it to approximate $\sqrt{17}$. Estimate the size of the error by using the error term of the Taylor polynomial you found.

8. Find the real and imaginary parts, the modulus $r=|z|$, the conjugate $\bar{z}$, and the argument θ of the following:

   (a) **[1 pt]** $z=7i$
   (b) **[1 pt]** $z=\sqrt{3}-i$
   (c) **[1 pt]** $z=3e^{i\frac{\pi}{4}}$
   (d) **[2 pts]** $z=\dfrac{\sqrt{2}}{i-1}$
   (e) **[3 pts]** $z=(1+i)^5$

9. **[3 pts]** Show the convergence or divergence of the sequence

$$a_n=\sqrt{\frac{2+\cos n}{n}} \qquad n=1,2,\ldots$$

10. Show absolute or conditional convergence or divergence for each of the following series.

    (a) **[3 pts]** $\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{\sqrt{n}}{n+2}$ (c) **[3 pts]** $\displaystyle\sum_{n=1}^{\infty}(-1)^n\left(\frac{1}{\sqrt{n}}-\frac{1}{\sqrt{n+1}}\right)$

    (b) **[3 pts]** $\displaystyle\sum_{n=2}^{\infty}\frac{1}{n\ln n}$

11. Find the radius of convergence of each of the series.

    (a) **[4 pts]** $\displaystyle\sum_{n=1}^{\infty}\frac{2^n x^n}{\sqrt{n}}$ (b) **[4 pts]** $\sum n!\,x^n$

12. (a) **[5 pts]** Find a power series representation for $f(x) = \displaystyle\int_0^x \frac{\sin t}{t}\, dt$

    (b) **[5 pts]** Use the power series obtained in (a) to approximate $f(1)$ accurate to two decimal places.