

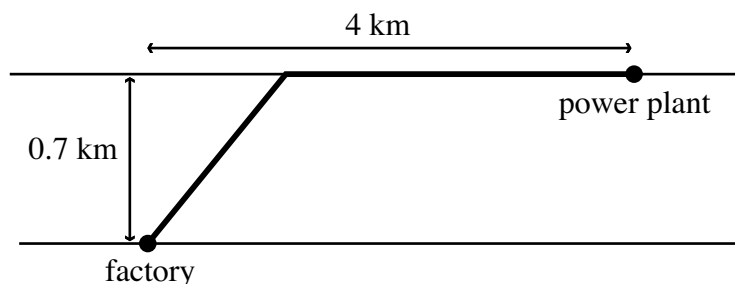
**University of Toronto at Scarborough
Physical Sciences Division, Mathematics**

**FINAL EXAMINATION
MATA26Y
Calculus**

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Date: April 23, 2002
Duration: 3 hours

1. (a) [4 pts] State the Mean Value Theorem
(b) [4 pts] Use the MVT to show that a continuous function $f(x)$ satisfying $f(4) = 1$ and $0 \leq f'(x) \leq \frac{3}{8}$ for $x \in (4, 12)$, must also satisfy $1 \leq f(x) \leq 4$ for $x \in [4, 12]$.
2. (a) [2 pts] Calculate the derivative of $f(x) = e^{\ln \tan x}$ at $x = \frac{1}{4}\pi$. Simplify your answer.
(b) [4 pts] Let $f(x) = x + \sin x$. Let g denote the inverse function of f . Calculate the derivative $g'(\frac{1}{4}\pi + \frac{1}{\sqrt{2}})$. [Hint: Note that $f(\frac{1}{4}\pi) = \frac{1}{4}\pi + \frac{1}{\sqrt{2}}$.]
3. [10 pts] A factory is located on one bank of a straight river that is 0.7 km wide. On the opposite bank, but downstream 4 km, is a power station which will supply electricity to the factory. If it is 4% cheaper to lay power lines above ground than underwater, what length of the cable should be laid underwater so as to minimize cost?



4. Calculate the integrals
 - (a) [4 pts] $\int \sin^3 u \cos^4 u \, du$
 - (b) [4 pts] $\int_0^1 \frac{x \, dx}{\sqrt{1-x^4}}$
5. [10 pts] Find the length of the graph of the curve $y = \frac{1}{2}(e^x + e^{-x})$ over the interval $-a \leq x \leq a$.

6. [10 pts] Determine the volume of the figure obtained by revolving, about the x -axis, the figure enclosed by the curves $y = x^2$ and $y = 8 - x^2$.
7. (a) [5 pts] Determine the 5th order Taylor polynomial for $\cos x$ about $a = 0$.
 (b) [5 pts] Determine the 5th order Taylor polynomial for $\frac{x}{1 - x^2}$ about $a = 0$.
8. (a) [2 pts] Express $\frac{1 + i}{2 + i}$ in rectangular form
 (b) [2 pts] Express $3 + 4i$ in polar form
 (c) [4 pts] Determine, in rectangular form, all complex numbers satisfying the equation $z^3 = 1$.
9. Consider the series $\sum_{k=0}^{\infty} a_k$
- (a) [2 pts] State the test for divergence of a series based on the limit of the k th term.
 (b) [2 pts] State the Limit Comparison Test for convergence.
 (c) [2 pts] State the Ratio Test for convergence.
 (d) [4 pts] State the Integral Test for convergence.
10. Determine whether each of the following series converges or diverges, giving your reasons why.
- (a) [2 pts] $\sum_{n=1}^{\infty} \frac{1 + n^4}{1 + n^2 + n^4}$
 (b) [2 pts] $\sum_{n=1}^{\infty} (-1)^n \frac{1 + n}{1 + n^2}$
 (c) [2 pts] $\sum_{n=1}^{\infty} (\sin n) \left(\frac{1}{2}\right)^n$
 (d) [2 pts] $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$
11. (a) [2 pts] If R is the radius of convergence of $\sum_{n=0}^{\infty} a_n(z - a)^n$, state the relationship between R and the convergence of the series.
 (b) [4 pts] Determine the radius of convergence of the series

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{(2n)!}{2^{2n}(n!)^2} x^n$$

(c) [6 pts] Find the interval of convergence of the series

$$\sum_{n=1}^{\infty} \frac{(3x-1)^n}{2n^8 - n^4}$$

12. [10 pts] **(Bonus Question)** Calculate the limit

$$\lim_{x \rightarrow 0} \frac{1 - \frac{1}{2}x^2 - \cos\left(\frac{x}{1-x^2}\right)}{x^4}$$