Contents

0 Mathematical Sign Language

Mathematical formulations should be precise and concise. To achieve this aim, mathematical language employs symbols and formulas. We review some basic notation in this chapter.

0.A Numbers

Certain sets of numbers have their own symbol.

$\mathbb{R}$ denotes the set of all *real numbers*,

$\mathbb{Q}$ denotes the set of all *rational* numbers, i.e. all fractions p/q with p, q integers and $q \neq 0$,

$\mathbb{Z}$ denotes the set of all *integers*,

$\mathbb{N}$ denotes the set of all *natural* numbers, these are the nonnegative integers. (Thus 0 is the smallest natural number.)

$\emptyset$ denotes the *empty* set, the only set that contains no element.

0.B Intervals

Intervals are special — and important! — subsets of the real numbers. They are solution sets to inequalities and come in various forms.

The **Open** interval (a, b) is the set of real numbers $\{x \in \mathbb{R} | a < x < b\}$,

The **Closed** interval $[a, b]$ is the set of real numbers $\{x \in \mathbb{R} | a \leq x \leq b\}$,

The **Half-open** interval $(a, b]$ is the set of real numbers $\{x \in \mathbb{R} | a < x \leq b\}$,

The **Half-open** interval $[a, b)$ is the set of real numbers $\{x \in \mathbb{R} | a \leq x < b\}$,

The **Infinite** interval $[a, \infty)$ is the set of real numbers $\{x \in \mathbb{R} | a \leq x\}$,

The **(Open) Infinite** interval (a, ∞) is the set of real numbers $\{x \in \mathbb{R} | a < x\}$,

The **Infinite** interval $(-\infty, b]$ is the set of real numbers $\{x \in \mathbb{R} | x \leq b\}$,

The **(Open) Infinite** interval $(-\infty, b)$ is the set of real numbers $\{x \in \mathbb{R} | x < b\}$,

The **Infinite** interval $(-\infty, \infty)$ is the same as the set $\mathbb{R}$ of all real numbers.

Warning:

1. The symbol ∞ is NOT a real number; ∞ is a notational convenience to indicate the interval is unbounded to the right, $-\infty$ unbounded to the left. To avoid this notation one sometimes writes $\mathbb{R}_{>a}$ for (a, ∞), $\mathbb{R}_{\geq a}$ for $[a, \infty)$, $\mathbb{R}_{<b}$ for $(-\infty, b)$ and $\mathbb{R}_{\leq b}$ for $(-\infty, b]$.

2. The symbol (a, b) is also used to denote a point in the plane. Thus it is necessary to write: "the solution is the interval (a, b)" or "the point is (a, b)".

3. Many sets of real numbers are not intervals, for example the integers, $\mathbb{Z} = \{\ldots -3, -2, -1, 0, 1, 2, 3 \ldots\}$. We will denote sets of real numbers that are not intervals by $\{a, b, c\}$, if we can list all of the elements, or we will describe them through a defining property, like $\mathbb{Q}_{>0} = \{x \in \mathbb{R} | x \text{ is rational and } x > 0\}$.

0.C Operations on Sets

A *set* is a collection of *elements*. Sets can be defined either by listing all elements, for example

$$S = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\},$$

or by describing the elements through characterizing properties

$$\begin{aligned} S &= \{\text{natural number from 1 upto and including 10}\} \\ &= \{x \in \mathbb{N} \mid 1 \leq x \leq 10\}. \\ T &= \{\text{real numbers that are zeros of } \sin(x)\} \\ &= \{x \in \mathbb{R} \mid x = n\pi \text{ for some } n \in \mathbb{Z}\}. \end{aligned}$$

One writes

$x \in S$ if x is an *element* in S, or shorter: *x is in S*.

$x \notin S$ if x is *not* an element from S, or: *x is not in S*.

For example, $0 \in \mathbb{N}$ means that zero is a natural number; $\pi \notin \mathbb{Q}$ means that $\pi = 3.1415926535\ldots$ is not a rational number.

One can *compare sets: If A, B are sets, then*

$A \subseteq B$ read: A is a *subset* of B,
means: *every* element from A is also an element of B.

$A = B$ read: A equals B,
means: A and B have precisely the *same* elements.

$A \subset B$ read: A is a *proper subset* of B,
means: A is a subset of B, but *not equal* to B.

One can form new sets from old ones:

$A \cup B$ read: *A union B*,
means: the set comprising all elements from A *or* from B.

$A \cap B$ read: *A intersect B*,
means: the set of all elements that are *both* in A *and* in B.

$A \setminus B$ read: *A minus B*,
means: the set of all elements that *are in* A but *not in* B.

$A \times B$ read: *A product B*,
means: the set of all *pairs* (a, b) where a is from A and b is from B.

One writes in particular

$\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ for the product of the set of real numbers with itself. The pairs $(x, y) \in \mathbb{R}^2$ are the *points* of the *real plane.*

0.D Implication and Equivalence

If S and T are *statements*, one writes

$S \Longrightarrow T$	read:	S *implies* T,
	means:	*if* S is true, *then* T is also true.
$S \Longleftrightarrow T$	read:	S *equivalent to* T, or: S if and only if T, or: S iff T,
	means:	S is true if and only if T is true.

Neither notation means that S or T is actually true! For example,

$(5 = 6) \Longrightarrow (1 = 2)$ is a *valid* or *true* implication:

> **If** 5 were equal to 6, **then** we could subtract 4 from both to obtain $1 = 2$.

Although both statements are false for real numbers, it is valid to conclude the second from the first.

$(5 = 6) \Longrightarrow (0 = 0)$ is also a *valid* implication:

> **If** 5 were equal to 6, **then** we could multiply both with 0 to obtain $0 = 0$.

The morale of this is:

> "From a false hypothesis, everything, even a true statement can be deduced",

—but you usually do not get credit for it!

> "(all Canadians are millionaires) $\Longrightarrow$ (all Canadians whistle all day)"

is *not* a *valid* implication: although it may be true, we have no way to *decide* the implication (today).

There are thus two aspects to both $\Longrightarrow$ or $\Longleftrightarrow$: It is one thing to know that whenever S is true, then T will be true, and it is another to actually know whether S *is* true. If we know that S is true *and* we know that $S \Longrightarrow T$ is *true/valid*, then we know also that T is true.

0.E Inequalities

Inequalities describe relations among real numbers. A real number is *positive* iff it is the square of another nonzero real number. In signs:

$$x > 0 \Longleftrightarrow \text{ there is a } y \in \mathbb{R} \text{ such that } y \neq 0 \text{ and } x = y^2.$$

We say that real numbers $a, b \in \mathbb{R}$ satisfy

$a < b$	read:	*a less than b*; or: *b larger than a*,
	means:	$b - a > 0$, i.e. $b - a$ is positive.
$a \leq b$	read:	*a less than* or *equal to b*; *a not larger than b*,
	means:	*either* $a < b$ *or* $a = b$.

Important properties of inequalities

(i) $(a < b \text{ and } b < c) \Longrightarrow (a < c)$, (Transitivity),

(ii) $(a < b \text{ and } c \text{ arbitrary}) \Longrightarrow (a + c < b + c)$,

(iii) $(a < b \text{ and } 0 < c) \Longrightarrow (ac < bc)$,

(iii') $(a < b \text{ and } c < 0) \Longrightarrow (bc < ac)$,

(iv) $(a < b) \Longleftrightarrow (-b < -a)$,

(v) For any $a \in \mathbb{R}$, $a^2 \geq 0$, and $a^2 > 0$ iff $a \neq 0$.

0.F Absolute Value

Definition: The *absolute value*, $|a|$, of a real number a is

$$|a| = \begin{cases} a & \text{if } a \geq 0, \\ -a & \text{if } a < 0. \end{cases}$$

Properties of absolute value:

1. $|x| \geq 0$; $|x| = 0$ if and only if $x = 0$.
2. $|xy| = |x||y|$; $|x/y| = |x|/|y|$ $(y \neq 0)$.
3. $|-x| = |x|$; $|x - y| = |y - x|$.
4. $|x|^2 = x^2$.
5. $\sqrt{x^2} = |x|$. (This is **very** important. Many commercial software packages have this wrong!)
6. If $a > 0$ then $|x| < a$ is equivalent to $-a < x < a$ or $x \in (-a, a)$.
7. If $a \geq 0$ then $|x| \leq a$ is equivalent to $-a \leq x \leq a$ or $x \in [-a, a]$.
8. If $a > 0$ then $|x - b| < a$ is equivalent to $b - a < x < b + a$ or x in the open interval $(b - a, b + a)$.

Note once again: $\sqrt{x^2} = |x|$.

The symbol $\sqrt{z}$ is defined for $z \geq 0$ and means the *positive* square root of the number z.

Also, if n is any *positive integer*, then

9. $|x| \leq 1 \Longrightarrow |x| \geq |x|^n$.
10. $|x| \geq 1 \Longrightarrow |x| \leq |x|^n$.

0.G Summation Notation

If a_i are numbers indexed by integers i then we write

$$a_m + a_{m+1} + \cdots + a_n = \sum_{i=m}^{n} a_i.$$

Examples.

1. $1 + 4 + 9 + 16 + 25 = \sum_{j=1}^{5} j^2.$

2. $2x + 3x^2 + 4x^3 + 5x^4 = \sum_{i=1}^{4} (i+1)x^i.$

3. $200 + 202 + 204 + 206 + 208 = \sum_{i=100}^{104} 2i = \sum_{k=0}^{4} (200 + 2k).$

Properties of Summation Notation:

1) $\sum_{j=m}^{n} a_i = \sum_{j=m}^{n} a_j = \sum_{\text{charly}=m}^{n} a_{\text{charly}}.$ (the naming of the summation variable is irrelevant)

2) $\sum_{i=m}^{n} (a_i + b_i) = \sum_{i=m}^{n} a_i + \sum_{i=m}^{n} b_i.$ (regrouping terms)

3) $\sum_{i=m}^{n} ba_i = b \sum_{i=m}^{n} a_i.$ (multiplication with a factor)

4) $\sum_{i=m}^{n} a_i = \sum_{i=m}^{p} a_i + \sum_{i=p+1}^{n} a_i$ where $m \leq p \leq n$. (breaking-up a sum)

5) $\sum_{i=m}^{m} a_i = a_m.$ (pretentious summation)

6) $\sum_{i=1}^{n} 1 = n, \quad \sum_{i=0}^{n} 0 = 0.$ (simple sums)

7) $\sum_{i=m}^{n} a_i = \sum_{i=m+p}^{n+p} a_{i-p}.$ (re-indexing)

8) $\sum_{i=1}^{n} (a_i - a_{i-1}) = a_n - a_0.$ (telescope)

0.H Factorial Notation

$n! = 1 \cdot 2 \cdot 3 \cdots (n-1)n$ if n is an integer greater than zero.

$0! = 1$, (this is a *convention*).

This definition makes the following identity valid for all positive integers.

$n! = (n-1)!n.$

$1! = 1, \quad 2! = 2, \quad 3! = 6, \quad 4! = 24, \quad 5! = 120, \quad 10! = 3628800 \approx 3.6 \cdot 10^6.$

0.I Binomial Coefficients

Case 1: n and k are integers, $0 \leq k \leq n$.

Define: $\binom{n}{k} = \dfrac{n(n-1)\cdots(n-k+1)}{1\cdot 2\cdot 3\cdots(k-1)k}$.

$\binom{n}{k} = \dfrac{n!}{k!(n-k)!}$, $\binom{n}{0} = 1$ since $0! = 1$.

Note: $\binom{n}{k}$ is the number of ways of choosing k objects out of a set of n objects, regardless of the order in which the k objects are chosen.

Properties:

1) $\binom{n}{k} = \binom{n}{n-k}$.

2) $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$.

Case 2: α is a real number, k an integer greater than or equal to 0.

Define:

$$\binom{\alpha}{k} = \tfrac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{k!}$$

$$\binom{\alpha}{0} = 1.$$

Example.

$\binom{\frac{1}{2}}{1} = \frac{1}{2}$, $\binom{\frac{1}{2}}{2} = -\frac{1}{8}$, $\binom{\frac{1}{2}}{3} = \frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right) = \frac{1}{16}$,

$\binom{\frac{1}{2}}{4} = \frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{24} = \frac{-5}{128}$.

The Binomial Theorem: For a positive integer n and $x, y \in \mathbb{R}$, we have:

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k.$$