Contents

II Polynomials and Rational Functions

Polynomials are perhaps the most important functions we will deal with. They are easy to manipulate or to evaluate, but finding roots of polynomials is one of the oldest and more difficult tasks in mathematics. We review the definitions and important properties; most of them you have probably seen before.

II.A Polynomials

A *real polynomial* is an expression of the form

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0 \qquad n \geq 0,\ n \in \mathbb{N}$$

with $a_i \in \mathbb{R}$ and x a variable. The a_i are called the *coefficients*, a_0 is the *constant coefficient.* If $a_n \neq 0$ we say $P(x)$ has *degree* n, deg $P = n$, and a_n is called the *leading coefficient* The *zero polynomial* $P(x) = 0$, i.e. $a_n = a_{n-1} = \cdots = a_0 = 0$ poses a bit of a problem: It has *no* degree, thus *no* leading coefficient, but constant coefficient zero! To exclude this pesty zero polynomial, we often speak only of "polynomials that have a degree" or of "nonzero polynomials".

If P and Q are nonzero polynomials, then

$$\deg(PQ) = \deg P + \deg Q,$$

$$\deg(P+Q)\begin{cases} = \max(\deg P, \deg Q) & \text{if } \deg P \neq \deg Q, \\ \leq \max(\deg P, \deg Q) & \text{if } \deg P = \deg Q. \end{cases}$$

If a_m is the leading coefficient of P and b_n the leading coefficient of Q, then $a_m b_n$ is the leading coefficient of $P \cdot Q$. The constant coefficient of $P \cdot Q$ is $a_0 b_0$ if a_0 is that of P and b_0 that of Q.

Examples.

1. The polynomial $P(x) = 2x + 1$ has degree 1, it is a *linear* polynomial. Its leading coefficient is 2, its constant coefficient is 1.
2. $P(x) = x^2 - \sqrt{5}x + 3$ is of degree 2, it is a *quadratic* polynomial.
3. $P(x) = \pi x^3 - x^2 + 2/9$ is of degree 3, it is a *cubic* polynomial.
4. $P(x) = 1124$ is of degree 0. It is a *constant* polynomial. The leading coefficient is 1124 which is also the constant coefficient.
5. If $P(x) = x^2 + 1$, $Q(x) = -x^2 + 1$, then deg P = deg $Q = 2$, and $(P \cdot Q)(x) = -x^4 + 1$ has degree 4, whereas $P(x) + Q(x) = 2$ has degree 0, less than $2 = \max(\deg P, \deg Q)$.

Polynomials are not always given in *expanded* form as above. For example, $P(x) = (x-1)(x+1)(x^2+1)$ is also a polynomial, of degree 4 with leading coefficient 1 and constant coefficient -1. Its expanded form is $P(x) = x^4 + 0 \cdot x^3 + 0 \cdot x^2 + 0 \cdot x - 1 = x^4 - 1$, as you can check easily.

A basic result on polynomials is that one can perform *polynomial* or *long division*:

Theorem I (Polynomial Division): Let $P(x), Q(x)$ be polynomials with deg $Q(x) = m$, (in particular, $Q(x)$ is not the zero polynomial!). There are then *unique* polynomials $S(x)$ and $R(x)$ such that

(i) $P(x) = S(x)Q(x) + R(x)$

and

(ii) either $R(x) = 0$ or deg $R(x) < m$.

The polynomial $R(x)$ is the *remainder*. The polynomial $Q(x)$ is a *factor* of $P(x)$ if and only if $R(x) = 0$.

Every polynomial may be interpreted as a real valued function with domain $\mathbb{R}$, by letting the variable x vary through the real numbers. A real number λ is a *zero* or *root* of $P(x)$ if $P(\lambda) = 0$. Graphically, $x = \lambda$ is an *x–intercept* of the graph of $P(x)$.

The following result allows one to find the "easy roots" of a polynomial — if there are any:

Theorem II (Rational Root theorem): If an equation $f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_0 = 0$, $a_n \neq 0$ with *integral* coefficients, i.e. $a_i \in \mathbb{Z}$, has a nonzero rational root p/q in its lowest terms then q is a divisor of a_n and p is a divisor of a_0.

The following applications of long division help to check for roots.

Theorem III (Remainder theorem): Let $P(x)$ denote a polynomial of degree $n \geq 1$ and c a real number. Then there exists a polynomial $S(x)$ of degree $n-1$ such that

$$P(x) = S(x)(x-c) + P(c).$$

Theorem IV (Factor theorem): Let $P(x)$ denote a polynomial of degree $n \geq 1$ and c a real number. Then $x - c$ is a factor of $P(x)$ if and only if $P(c) = 0$, that is, if and only if c is a root of $P(x)$.

Proof of theorems III and IV: With $Q(x) = x - c$, long division yields an equation

$$P(x) = S(x)(x-c) + R(x).$$

As $x - c$ is of degree 1, the remainder is either of degree 0 or zero. Thus $R(x) = a$ for some real number $a \in \mathbb{R}$, so that

$$P(x) = S(x)(x-c) + a.$$

What is a? Substituting $x = c$, we find

$$P(c) = S(c) \cdot 0 + a = a,$$

thus a is the value of P at c as claimed in theorem III. Furthermore, $P(c) = 0$ iff $a = 0$ iff $x - c$ is a factor of $P(x)$, whence theorem IV. □

In view of theorem III, the constant coefficient of a polynomial is the same as its value at $x = 0$, the y–intercept of the graph of P.

Theorem V: Every nonzero polynomial $P(x)$ can be written

$$P(x) = (x - \lambda_1)(x - \lambda_2) \cdots (x - \lambda_r)Q(x)$$

where $\lambda_1, \ldots, \lambda_r$ are the zeros of $P(x)$ — possibly repeating — and $Q(x)$ has no zeros. In particular, $P(x)$ has at most n zeros if deg $P(x) = n$.

Proof: We prove this by induction on the degree, n, of $P(x)$. We have to check that

(a) The theorem is true when $n = 0$.

(b) If the theorem is true for all polynomials of some degree $n \geq 0$, then it is true for all polynomials of degree $n + 1$.

Verifying (a). When $n = 0$, $P = a_0$ is a constant and $a_0 \neq 0$ as P has a degree. Thus P itself has no zeros and we set $Q(x) = P(x)$.

Verifying (b). Suppose $P(x)$ has degree $n + 1$. If $P(x)$ has no zeros we set $Q(x) = P(x)$.

If $P(x)$ has a zero, λ, then we can write (by theorem IV)

$$P(x) = (x - \lambda)P_1(x).$$

Now $P_1(x)$ has degree n, and so we are assuming the theorem is true for $P_1(x)$. Thus

$$P_1(x) = (x - \lambda_1) \cdots (x - \lambda_r)Q(x)$$

and $Q(x)$ has no zeros. Substitute this in the equation above:

$$\begin{aligned} P(x) &= (x - \lambda)P_1(x) \\ &= (x - \lambda)(x - \lambda_1) \cdots (x - \lambda_r)Q(x). \end{aligned}$$

This shows that theorem V is true for polynomials P of degree $n + 1$, as soon as it is true for polynomials of degree n. The principle of induction shows now that the theorem is true for every polynomial that has a degree; in other words, it holds for every nonzero polynomial. □

Corollary: If $\lambda_1 \neq \lambda_2$, then $P(x)$ is divisible by $(x - \lambda_1)(x - \lambda_2)$ iff λ_1, λ_2 are roots of $P(x)$, i.e. $P(\lambda_1) = P(\lambda_2) = 0$.

Example. $(x^2 - 1)|P(x) = x^5 - x^4 - x + 1$ as $x^2 - 1 = (x - 1)(x + 1)$ and $P(1) = P(-1) = 0$. Indeed, $P(x) = (x - 1)(x - 1)(x + 1)(x^2 + 1)$ so that $\deg P = 5$, $Q(x) = x^2 + 1$ has no root, $r = \#$ of roots $= 3$, and the roots of $P(x)$ are $1, 1, -1$.

Theorem VI: Suppose $P(x)$ and $Q(x)$ are both polynomials with $R(x) = P(x) - Q(x)$ a (nonzero!) polynomial of degree $\leq n$. Then the equation

$$P(x) = Q(x)$$

has at most n solutions. In particular, P and Q are different functions.

Proof: $R(x) = P(x) - Q(x)$ is a non-zero polynomials of degree at most n. Thus $R(x)$ has at most n roots. But

$$R(x) = 0 \Leftrightarrow P(x) = Q(x) \quad \text{for every } x \in \mathbb{R}.$$ □

Theorem VI restated: Suppose $P(x)$ and $Q(x)$ take the same values at $n + 1$ distinct real numbers $\lambda_1 \ldots, \lambda_{n+1}$:

$$P(\lambda_i) = Q(\lambda_i) \qquad \text{for } 1 \leq i \leq n + 1,$$

and suppose further that $\deg P \leq n$ and $\deg Q \leq n$. Then P and Q are the same polynomials.

Theorem V can be sharpened considerably — that is the content of the FUNDAMENTAL THEOREM OF ALGEBRA in its real form.

To formulate it, recall the following classification of real quadratic polynomials. The statement you certainly remember from school is the quadratic formula:

The roots of $P(x) = ax^2 + bx + c$ are

$$x_{1,2} = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a} \qquad \textit{provided} \quad a \neq 0 \textit{ and } b^2 - 4ac \geq 0.$$

Another way to state it is as follows.

Theorem VII (Structure of Quadratic Polynomials): If $P(x) = ax^2 + bx + c$ is a quadratic polynomial with $a, b, c \in \mathbb{R}$ and $a \neq 0$ then there are real numbers α, β such that

$$P(x) = a(x + \alpha)^2 + \beta.$$

If a and β have the *same sign*, i.e. $a\beta > 0$, then $P(x)$ has no root. If $a\beta \leq 0$, then $-\frac{\beta}{a} \geq 0$, the square root $\sqrt{-\frac{\beta}{a}}$ exists in $\mathbb{R}$ and $P(x) = a\left(x + \alpha + \sqrt{-\frac{\beta}{a}}\right)\left(x + \alpha - \sqrt{-\frac{\beta}{a}}\right)$.

Proof: Take $\alpha = \frac{b}{2a}$ and $\beta = c - \frac{b^2}{4a}$. □

The real number $\triangle := b^2 - 4ac = -4a\beta$ is the *discriminant* of the quadratic polynomial as it discriminates between two possibilities:

if $\triangle \geq 0$, then $a\beta \leq 0$ and $P(x)$ has *two* roots (*distinct*, if $\triangle > 0$; *equal* if $\triangle = 0$),

if $\triangle < 0$, then $a\beta > 0$ and $P(x)$ has *no* roots.

The process of writing $P(x) = a(x + \alpha)^2 + \beta$ is also called *completing the square.*

A quadratic polynomial without roots, hence with negative discriminant, is also called *irreducible*, as it cannot be reduced to a product of linear polynomials.

The real form of the Fundamental Theorem of Algebra is now easy to state.

Theorem VIII (Fundamental Theorem of Algebra): Every nonzero polynomial $P(x)$ can be written as a product of linear polynomials and of irreducible quadratic polynomials,

$$P(x) = (x - \lambda_1) \cdots (x - \lambda_r) \left(a_1(x + b_1)^2 + c_1\right) \left(a_2(x + b_2)^2 + c_2\right) \cdots \left(a_s(x + b_s)^2 + c_s\right)$$

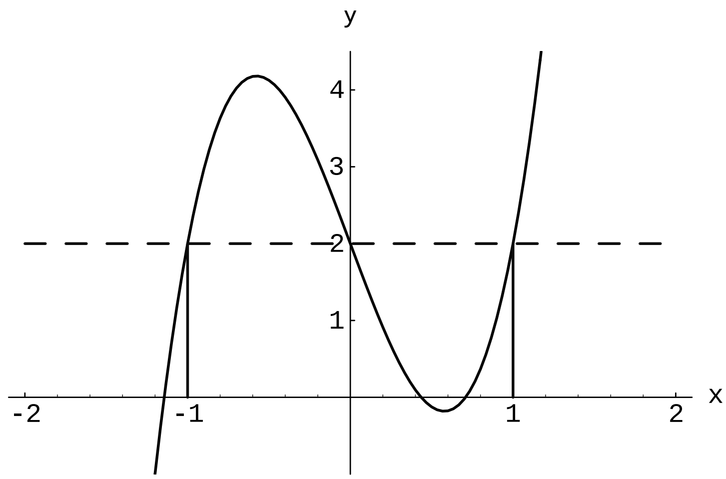
where $\lambda_1, \ldots, \lambda_r$ are the roots of $P(x)$ and $a_1c_1, \ldots, a_sc_s > 0$. As with the roots, some of the factors $a_i(x + b_i)^2 + c_i$ may be repeating. □

C.F. Gauss (1777–1855) was the first to prove the Fundamental Theorem of Algebra, and he was so proud of it that he gave several different proofs (some sources say 26) during his lifetime!

The big problem with the theorem is that it is not constructive: For polynomials of high degree (≥ 5 to be precise), there is no algorithm or recipe how to find either the roots or the quadratic factors exactly.

Example. The polynomial $P(x) = x^4 + 1$ has certainly no roots as $x^4 \geq 0$ for all x and thus $P(x) \geq 0 + 1 = 1$ for all x. But according to theorem VIII we should be able to write

$$x^4 + 1 = \left(a_1x^2 + b_1x + c_1\right)\left(a_2x^2 + b_2x + c_2\right).$$



Pulling out a_1, a_2 and comparing coefficients of x^4, we try

$$x^4 + 1 = (x^2 + bx + c)(x^2 + dx + e).$$

If we *assume* (a really wild guess and nothing else!) that $c = e = 1$, we obtain after expanding the right hand side

$$x^4 + 1 = x^4 + (b + d)x^3 + (bd + 2)x^2 + (b + d)x + 1.$$

As there are no terms x^3, x at the left, we have to have $b + d = 0$, i.e. $d = -b$ and as there is no x^2 either, we also need $-b^2 + 2 = 0$, whence $b = \pm\sqrt{2}$ and then $c = \mp\sqrt{2}$. Now check it to see that indeed

$$x^4 + 1 = (x^2 + \sqrt{2}\,x + 1)(x^2 - \sqrt{2}\,x + 1) = \left((x + \frac{1}{\sqrt{2}})^2 + \frac{1}{2} \right) \left((x - \frac{1}{\sqrt{2}})^2 + \frac{1}{2} \right).$$

Were we lucky!

One can translate algebraic properties of polynomials into geometry and vice versa.

Example. Find a cubic polynomial that has the following graph

Solution. If $p(x) = ax^3 + bx^2 + cx + d$ is our mystery polynomial, we have to have $p(-1) = p(0) = p(1) = 2$.

This means that $q(x) = p(x) - 2$ has the roots $-1, 0, 1$. As $q(x)$ is also at most of degree 3 (why?), we get $q(x) = (x - (-1))(x - 0)(x - 1) \cdot Q(x)$ where $Q(x)$ is a polynomial of degree 0, thus a constant. Hence

$$q(x) = (x + 1)x(x - 1) \cdot a = a(x^3 - x),$$

and so

$$p(x) = q(x) + 2 = a(x^3 - x) + 2.$$

As $p(x)$ is positive for $x \geq 1$ and the values $p(x)$ become larger with larger x's, we can also conclude $a > 0$. But that is all, any polynomial

$$p(x) = a(x^3 - x) + 2 \quad \text{for } a > 0$$

will do — try various positive values of a on your calculator.

II.B Rational Functions

An expression of the form

$$f(x) = \frac{P(x)}{Q(x)}, \quad P, Q \text{ polynomials, } Q \text{ not the zero polynomial,}$$

is called a *rational function.* We think of it as a real valued function with *domain*

$$\text{dom}(f) := \text{domain } f = \{x \in \mathbb{R} | Q(x) \neq 0\},$$

and *range*

$$\text{range(f)} := \text{f (dom(f))} = \{\text{y} \in \mathbb{R} \mid \text{y} = \text{f(x) for some } z \in \text{dom}(f)\}.$$

A real number λ is a *zero* or *root* of f if

$$\lambda \in \text{domain } f \quad \text{and} \quad f(\lambda) = 0.$$

Property 1: If $f(x) = P(x)/Q(x)$ and degree $P \leq n$ then $f(x)$ has at most n zeros.

Proof: If $f(\lambda) = 0$ then $P(\lambda) = 0$. Apply theorem V for polynomials. □

Property 2: If $f(x) = P(x)/Q(x)$ and degree $Q \leq m$ then $f(x)$ is undefined for at most m real numbers.

Proof: The numbers where f is undefined are the zeros of Q. There are at most m of these. □

Property 3: The sum, product and quotient (∗) of rational functions are rational functions.

Proof:

$$\frac{P_1}{Q_1} + \frac{P_2}{Q_2} = \frac{P_1Q_2 + Q_1P_2}{Q_1Q_2}.$$

$$\frac{P_1}{Q_1} \cdot \frac{P_2}{Q_2} = \frac{P_1P_2}{Q_1Q_2}.$$

$$\left(\frac{P_1}{Q_1}\right) \Big/ \left(\frac{P_2}{Q_2}\right) = \frac{P_1Q_2}{P_2Q_1}. \quad \text{((*) One needs that } P_2 \text{ is not the zero function!)}$$

□

Examples

1. The rational function
$$f(x) = \frac{x^2 - 4}{(x^2 + 1)x} = \frac{(x - 2)(x + 2)}{(x^2 + 1)x}$$
has domain $\mathbb{R} - \{0\}$ and zeros $2, -2$.
2. The rational function
$$f(x) = \frac{x^2}{x}$$
has domain $\mathbb{R} - \{0\}$, no zeros, and range $\mathbb{R} - \{0\}$.
3. Trick question: What are the zeros of $f(x) = \frac{0}{x(x-1)(x-2)}$?

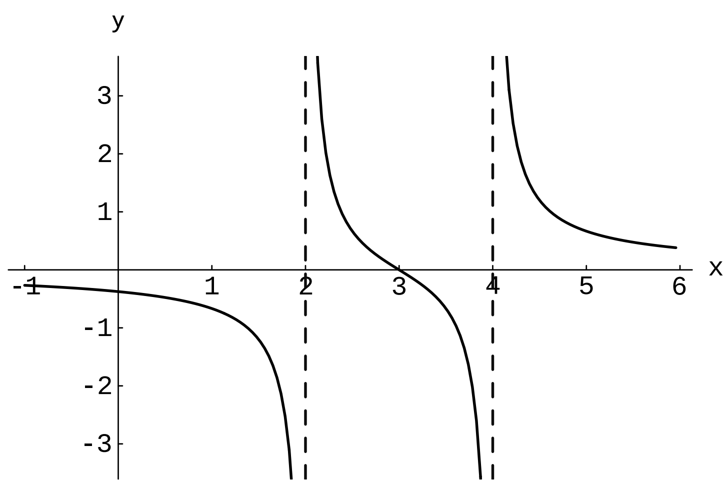
   Answer: All real numbers different from 0, 1, 2.

II.C Graph of a Rational Function

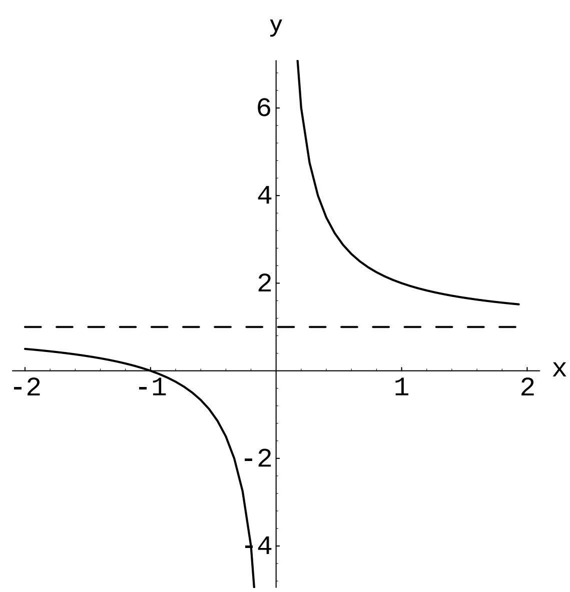
The *graph* of a rational function f is the set of points (x, y) in the plane such that $x \in \mathrm{dom}(f)$, $y = f(x)$.

Examples.

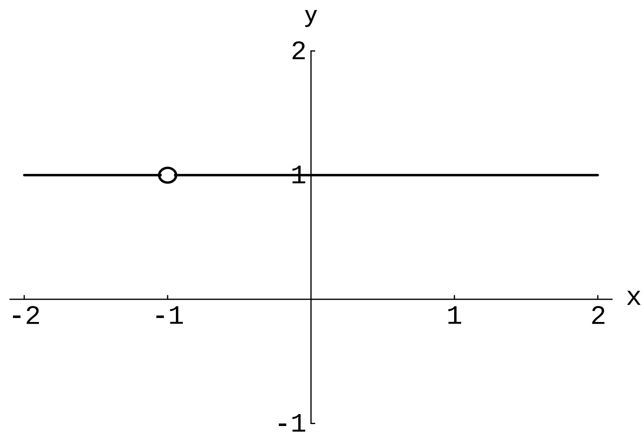
1. $f(x) = \frac{x-3}{x^2-6x+8} = \frac{x-3}{(x-2)(x-4)}$.
   The domain is $\mathrm{dom}(f) = \mathbb{R} - \{2, 4\}$
   There is one zero at 3.



2. $f(x) = \frac{x+1}{x} = 1 + \frac{1}{x}$
   domain: $\mathbb{R} - \{0\}$
   range: $\mathbb{R} - \{1\}$.



3. $f(x) = \frac{x+1}{x+1}$
   domain: $\mathbb{R} - \{-1\}$
   range: $\{1\}$.



II.D The Intermediate Value Theorem (IVT) for Rational Functions

We often need to study functions not on all of the domain but just on some interval. If $[a, b]$ is an interval contained in the domain of a function f, the *image* of $[a, b]$ under f is

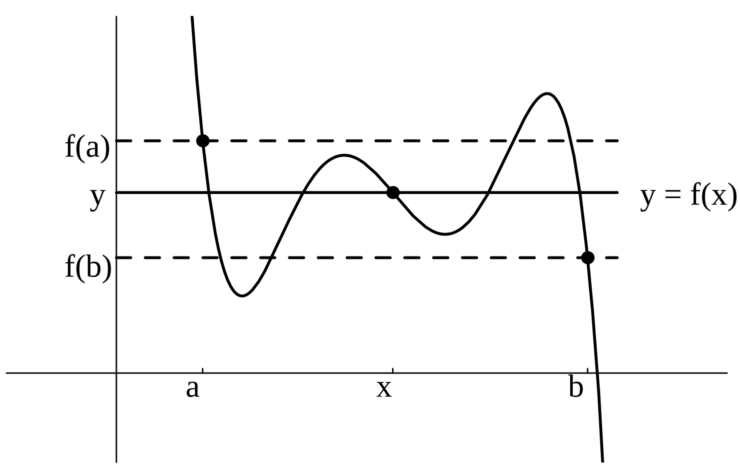
$$f([a, b]) = \{y \in \mathbb{R} \mid y = f(x) \text{ for some } x \in [a, b]\}.$$

One can talk as well of images of (half-) open intervals or other sets. The range of f is thus the same as the image of the domain of f under f.

Theorem: (IVT for rational functions)

Let $f(x) = P(x)/Q(x)$ be a rational function defined for all x in a closed interval $[a, b]$, i.e. $[a, b] \subseteq \text{dom}(f)$. Then as x varies from a to b, $f(x)$ takes on every value between $f(a)$ and $f(b)$.

In other words, the image, $f([a, b])$, of $[a, b]$ under f contains the closed interval bounded by $f(a)$ and $f(b)$.



In the picture every y between $f(a)$ and $f(b)$ is equal to some $f(x)$, $x \in [a, b]$.

Notes:

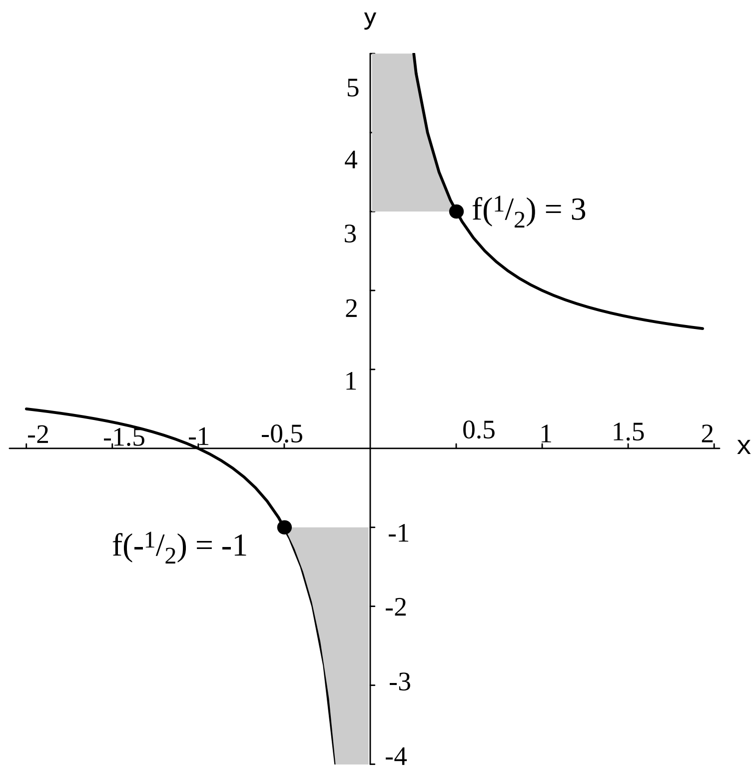
a) $f(x)$ might take on other values as well, for example in the above picture values larger than $f(a)$ are taken on.

b) A value between $f(a)$ and $f(b)$ can be taken on *several* times, as happens for the value y indicated in the picture. IVT assures us only that each value between $f(a)$ and $f(b)$ is taken *at least once*.

The following analogy is a good way to remember IVT:

If a mountain climber wants to get from "height" $f(a)$ to "height" $f(b)$, she has to pass every height in between at least once — assuming that there is some path from $f(a)$ to $f(b)$ and no abyss in between!

Remark: If there is an "abyss" between a and b, then $[a, b]$ is not contained in the domain of f, and IVT can fail. For instance in the case $f(x) = \frac{x+1}{x}$:



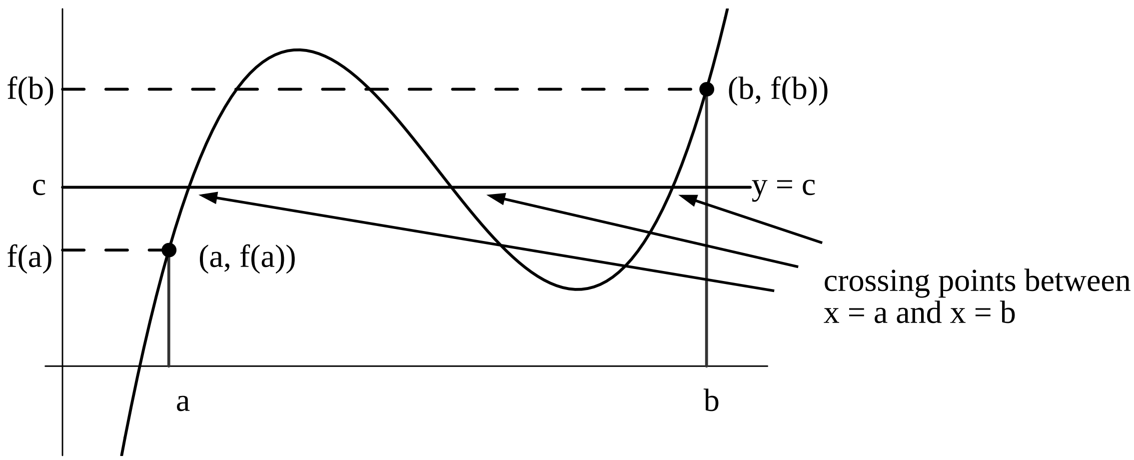
We have:

- $f(-\frac{1}{2}) = -1$; $f(\frac{1}{2}) = 3$.
- 1 is a number between -1 and 3.
- In the closed interval $[-\frac{1}{2}, \frac{1}{2}]$ there is no x such that $f(x) = 1$. $\left(f(x) = 1 + \frac{1}{x}\right)$

The point is: $0 \in [-\frac{1}{2}, \frac{1}{2}]$ and $0 \notin$ domain f. Thus $[-\frac{1}{2}, \frac{1}{2}]$ is not contained in the domain of f.

Another way of stating the IVT for rational functions

Let $f(x)$ be a rational function defined at each point of $[a, b]$, and let c be any real number between $f(a)$ and $f(b)$. Then the graph of f crosses the line $y = c$ between $x = a$ and $x = b$. (It may happen more than once!)



Proof of IVT from the Fundamental Theorem of Algebra:

Step 1: We first show that the following form of IVT holds for polynomials: If $P(x)$ is a polynomial such that $P(a)P(b) < 0$, i.e. $P(a)$ and $P(b)$ have *opposite sign*, then P has a root in (a, b). We prove this by CONTRADICTION: Assume the statement is not true, so that there is no root in (a, b). Writing $P(x) = (x - \lambda_1) \cdots (x - \lambda_r) \left(a_1(x + b_1)^2 + c_1\right) \cdot \left(a_s(x + b_s)^2 + c_s\right)$ according to the Fundamental Theorem of Algebra, we may then relabel the roots such that

$$\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_{r'} < a < b < \lambda_{r'+1} \leq \cdots \leq \lambda_r,$$

as there is supposed to be no root in (a, b) and neither a nor b can be a root, due to the assumption $P(a)P(b) < 0$. But then for every $z \in [a, b]$ one has

$$\begin{array}{lll} (z - \lambda_i) > 0 & \text{for} & 1 \leq i \leq r' \qquad \text{and} \\ (z - \lambda_j) < 0 & \text{for} & r' + 1 \leq j \leq r. \end{array}$$

As for the irreducible quadratic factors, the sign of $a_k(z + b_k)^2 + c_k$ equals the sign of a_k— which equals also the sign of c_k as $a_k c_k > 0$ — and this holds for *any* real z, not only for $z \in [a, b]$. Putting it together, each factor of $P(x)$ has *constant sign* on $[a, b]$, thus $P(a)$ and $P(b)$ in particular have the same sign, contradicting our hypothesis.

Step 2: If now $P(x)$ is some polynomial, c some value strictly between $P(a)$ and $P(b)$, then $(P(a) - c)\,(P(b) - c) < 0$:

$$\begin{array}{llll} \text{either} & P(a) < c < P(b) & \text{and then} & P(a) - c < 0 < P(b) - c \\ \text{or} & P(a) > c > P(b) & \text{and then} & P(a) - c > 0 > P(b) - c, \end{array}$$

so in both cases, $P(a) - c$ and $P(b) - c$ have opposite sign, i.e. $(P(a) - c)\,(P(b) - c) < 0$.

Now apply Step 1 to $P(x) - c$. This new polynomial changes sign on $[a, b]$ and has thus a root $z \in (a, b)$. But then $P(z) - c = 0 \Longleftrightarrow P(z) = c$ and we obtain IVT for polynomials.

Step 3: If $f(x) = \frac{P(x)}{Q(x)}$ is a rational function whose domain includes $[a, b]$, then $Q(x)$ has no root in $[a, b]$. Thus, as seen in Step 1, $Q(x)$ has the same sign for each $x \in [a, b]$. In particular, $Q(a)Q(b) > 0$. If now c is a value strictly between $f(a)$ and $f(b)$, then again

$$(f(a)-c)(f(b)-c)<0 \iff \left(\frac{P(a)}{Q(a)}-c\right)\left(\frac{P(b)}{Q(b)}-c\right)<0$$

$$\iff (P(a)-cQ(a))(P(b)-cQ(b))<0, \quad \text{as } Q(a)Q(b)>0.$$

The polynomial $R(x)=P(x)-cQ(x)$ changes sign on $[a,b]$, as it has opposite signs at a and b, so it has a root $z\in(a,b)$. But

$$R(z)=0 \iff P(z)-cQ(z)=0 \iff \frac{P(z)}{Q(z)}-c=0 \quad (\text{as } Q(z)\neq 0)$$

$$\iff f(z)-c=0 \iff f(z)=c.$$

We have thus established IVT for any rational function. □

The crucial result from Step 1 in the proof above is worthwhile to remember and we state it explicitly once more.

Corollary of the Constant Sign (CCS): Let $[a,b]$ be a closed interval in the domain of a rational function $f(x)$. Assume that

$$f(x)\neq 0, \quad x\in[a,b].$$

Then $f(x)$ has the same sign in $[a,b]$: either $f(x)>0$, for *all* $x\in[a,b]$ or $f(x)<0$, for *all* $x\in[a,b]$.

Proof: By the IVT, the graph of f is either entirely above, or else entirely below the line $y=0$ (x-axis) between $x=a$ and $x=b$. □