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III The Mean Value Theorem for Rational Functions

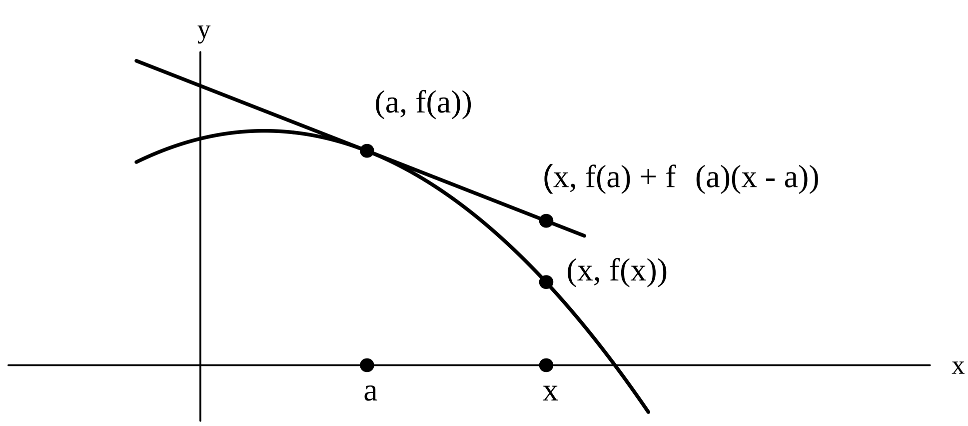
III.A Derivatives of Rational Functions and Tangent Lines

Without any geometry, one may define derivatives of rational functions in a purely algebraic way:

- The derivative of x^n is $(x^n)' = nx^{n-1}$.
- The derivative of a polynomial
$$P(x) = \sum_{i=0}^{n} a_i x^i$$
is given by
$$P'(x) = \sum_{i=1}^{n} i a_i x^{i-1}.$$
- The derivative of a rational function $f(x) = P(x)/Q(x)$ is given by
$$f'(x) = \frac{P'(x)Q(x) - P(x)Q'(x)}{(Q(x))^2}.$$
$f'(x)$ is again a rational function and $\mathrm{dom}(f') = \mathrm{dom}(f)$.
- If $f(x)$, $g(x)$ are rational functions and $a, b \in \mathbb{R}$ then
$$\begin{aligned}
[af(x) + bg(x)]' &= af'(x) + bg'(x) \\
[f(x)g(x)]' &= f'(x)g(x) + f(x)g'(x) && \text{(product rule)} \\
[f(x)/g(x)]' &= \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2} && \text{(quotient rule, } g(x) \text{ not the zero polynomial)}
\end{aligned}$$

The geometric interpretation of the derivative is through the *slope of the tangent line:*

Definition: If $f(x)$ is a rational function defined at $a \in \mathbb{R}$, then the *tangent line* to the graph of f at $(a, f(a))$ has equation
$$y = f(a) + f'(a)(x - a).$$
It is the *unique line* that passes through the point $(a, f(a))$ and that has slope $f'(a)$.



Once more:

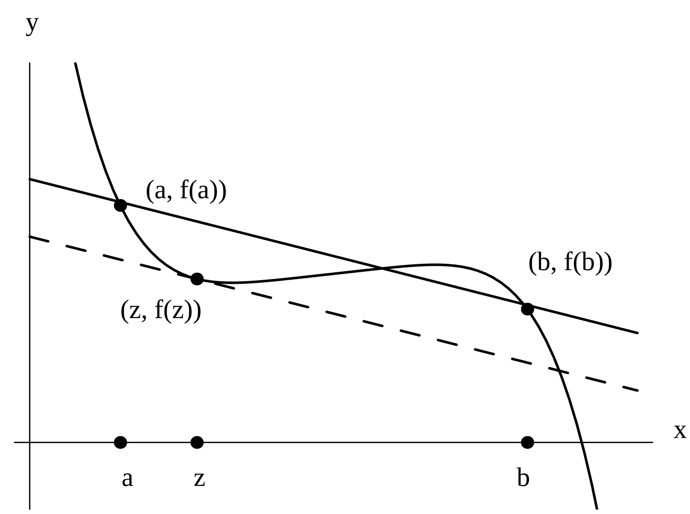
- the slope of the tangent line through $(a, f(a))$ is $f'(a)$.
- the equation of the tangent line is $y = f(a) + f'(a)(x - a)$.

III.B Mean Value Theorem, or MVT, and Rolle's Theorem for Rational Functions

The Mean Value Theorem (MVT for rational functions): Let $f(x) = P(x)/Q(x)$ be a rational function defined at each point of a closed interval $[a, b]$. Then for *some* $z \in (a, b)$:

$$f'(z) = \frac{f(b) - f(a)}{b - a}.$$

By assumption, the rational function, f, is defined at each point of $[a, b]$. Then



- The straight line through $(a, f(a))$ and $(b, f(b))$ has slope
$$\frac{\triangle f(x)}{\triangle x} = \frac{f(b) - f(a)}{b - a},$$
the *average rate of change of* f *over* $[a, b]$.
- MVT says that this is the same as the *instantaneous rate of change*, the derivative $f'(z)$, for some $z \in (a, b)$.
- The derivative, $f'(z)$, is the slope of the tangent line through $(z, f(z))$.
- Thus the MVT states: the slope of the line joining $(a, f(a))$ to $(b, f(b))$ is the same as the slope of the tangent line at some point $(z, f(z))$ with z between a and b.

As with IVT, the Mean Value Theorem *does not* predict *where* z will be found in (a, b), nor does it say that there is a unique such z.

In the picture above there are at least two points in (a, b) that can serve as a solution to $f'(z) = \frac{f(b)-f(a)}{b-a}$. Can you spot a second solution?

Example.

$f(x) = x^3 - 2x^2 + 3x + 1$; $a = 0$; $b = 2$.

$f(a) = 1,\ f(b) = 8 - 8 + 6 + 1 = 7$; $b - a = 2$.

Therefore, the average rate of change on $[0, 2]$ is $\frac{f(b)-f(a)}{b-a} = \frac{7-1}{2} = 3$.

The MVT asserts thus that for some $z \in (0, 2)$, $f'(z) = 3$.

In this simple case, we can check this — and actually *find* such a z:

$f'(x) = 3x^2 - 4x + 3$, and to solve $f'(z) = 3$ we need

$$\begin{aligned} 3z^2 - 4z + 3 = 3 \quad &\Rightarrow \quad 3z^2 - 4z = 0 \\ &\Rightarrow \quad z = 0 \quad \text{or} \quad z = \frac{4}{3} \end{aligned}$$

$z = 0$ *will not* do: $0 \notin (0, 2)$.

$z = \frac{4}{3}$ *will* do: $\frac{4}{3} \in (0, 2)$.

Suppose in the MVT that $f(a) = f(b)$. Then the RHS of the equation is zero and the conclusion is: for some $z \in (a, b)$, $f'(z) = 0$. This special case is very important and has its own name:

Rolle's Theorem (for rational functions): Let f be a rational function defined at each point of $[a, b]$ and suppose $f(a) = f(b)$. Then for some $z \in (a, b)$, $f'(z) = 0$.

Proof of Rolle's theorem: Replacing f by $f - d$, where $d = f(a) = f(b)$, we may suppose that $f(a) = f(b) = 0$: the theorem holds for f iff it holds for $f - d$ as $f' = (f - d)'$.

Step 1: We know that f takes the value zero at only finitely many points. Let c be the smallest number such that $a < c$ and $f(c) = 0$. (Either $c = b$ or $a < c < b$.) Thus we may assume

- $f(a) = 0,\ f(c) = 0$.
- f is defined in all of $[a, c]$.
- $f(x) \neq 0$ if $x \in (a, c)$.

Step 2: Write $f(x) = P(x)/Q(x)$ where P and Q are polynomials. Since f is defined in all of $[a, c]$,

- $Q(x) \neq 0,\ x \in [a, c]$.

Since $f(a) = 0$ and $f(c) = 0$,

- a and c are roots of $P(x)$.

Since $f(x) \neq 0,\ x \in (a, c)$,

- P has no roots between a and c.

Step 3: Since $P(x)$ is a polynomial with roots at a and c, we can write

$$P(x) = (x-a)^k (x-c)^\ell R(x)$$

where $k \geq 1$, $\ell \geq 1$ and $R(a) \neq 0$, $R(c) \neq 0$. Since $P(x) \neq 0$, $a < x < c$ we have

$$R(x) \neq 0, \qquad a < x < c.$$

Therefore $R(x) \neq 0$, $a \leq x \leq c$.

Step 4: Let $g(x) = R(x)/Q(x)$. Because $Q(x) \neq 0$, $a \leq x \leq c$, $g(x)$ is defined in all of $[a, c]$. Because $R(x) \neq 0$, $a \leq x \leq c$, $g(x)$ is never zero on $[a, c]$. Therefore by the Corollary of the constant sign,

- $g(a)$ and $g(c)$ have the same sign.

Step 5:

$$\begin{aligned} f(x) &= \frac{P(x)}{Q(x)} = \frac{(x-a)^k (x-c)^\ell R(x)}{Q(x)} \\ &= (x-a)^k (x-c)^\ell \cdot \frac{R(x)}{Q(x)} \\ &= (x-a)^k (x-c)^\ell g(x) \\ \text{Differentiating gives } f'(x) &= k(x-a)^{k-1}(x-c)^\ell g(x) + (x-a)^k \ell (x-c)^{\ell-1} g(x) + (x-a)^k (x-c)^\ell g'(x) \\ &= (x-a)^{k-1}(x-c)^{\ell-1}[k(x-c)g(x) + \ell(x-a)g(x) + (x-a)(x-c)g'(x)]. \end{aligned}$$

We define a new rational function, $h(x)$, by

$$h(x) = k(x-c)g(x) + \ell(x-a)g(x) + (x-a)(x-c)g'(x)$$

so that

- $f'(x) = (x-a)^{k-1}(x-c)^{\ell-1} h(x)$.

Step 6: Note that h is defined in all of $[a, c]$. Moreover

$$\begin{aligned} h(a) &= k(a-c)g(a) \\ h(c) &= \ell(c-a)g(c). \end{aligned}$$

By Step 4, $g(a)$ and $g(c)$ have the same sign.

Therefore $h(a)$ and $h(c)$ have opposite signs.

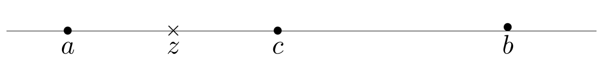
Therefore by IVT, for some $z \in (a, c)$,

$$h(z) = 0.$$

Step 7 (and last): By Step 4,

$$f'(z) = (z-a)^{k-1}(z-c)^{\ell-1} h(z) = 0.$$

Therefore $f'(z) = 0$ for some z between a and c. But since either $c = b$ or c is between a and b, z is in any case between a and b:



□

Proof of the MVT for rational functions: Suppose $f(x)$ is a rational function defined in all of $[a,b]$ and consider the rational function

$$F(x) = f(x) - f(a) - \left[\frac{f(b)-f(a)}{b-a}(x-a)\right]$$

which is also defined in $[a,b]$. Then

$$F(a) = 0 \quad \text{and} \quad F(b) = 0.$$

Therefore by Rolle's theorem, there is some $z \in (a,b)$ such that

$$F'(z) = 0.$$

However

$$F'(x) = f'(x) - \frac{f(b)-f(a)}{b-a}.$$

Therefore

$$0 = F'(z) = f'(z) - \frac{f(b)-f(a)}{b-a};$$

i.e.

$$f'(z) = \frac{f(b)-f(a)}{b-a}.$$

□

The following reformulation of Rolle's theorem is often used.

Corollary: Let f be a rational function defined on $[a,b]$. If $f'(x) \neq 0$ for $x \in (a,b)$, then f is *injective* (or 1-1) on $[a,b]$. Thus every value that f takes on is taken on exactly once.

Proof: If $f(c) = f(d)$ for some $c, d \in [a,b]$ and $c \neq d$, then f is defined on $[c,d] \subseteq [a,b]$ and Rolle's theorem says that f' has a root in (c,d). As f' has no root in the larger interval $[a,b]$, we have a contradiction and so there is no pair $c, d \in [a,b]$ with $c \neq d$ and $f(c) = f(d)$. □

Example. If P is a polynomial such that $P'(x) \neq 0$ for $x \in [a,b]$, then P has *at most one root* in (a,b): If the value 0 is taken on at all, i.e. P has a root, it is not taken on more than once, thus there is at most one root.

III.C Application of the MVT

We can use MVT to solve our first differential equation.

Proposition: If $f(x)$ is a rational function defined everywhere on an interval $[a,b]$ and if

$$f'(x) = 0, \qquad x \in (a,b)$$

then f is constant in $[a,b]$.

Proof: If x is any point in $(a,b]$ then for some $z \in (a,x)$:

$$\frac{f(x)-f(a)}{x-a} \underset{\text{MVT}}{=} f'(z) = 0 \qquad \text{(by hypothesis)}$$

and so

$$f(x) = f(a), \qquad a \le x \le b. \qquad \square$$

In the parlance of differential equations, constants $y = c$ are the only solutions in rational functions to the equation $y' = 0$ on $[a, b]$.

III.D Extended Mean Value Theorem (EMVT) for Rational Functions

The Extended Mean Value Theorem (EMVT for rational functions): Let $f(x)$ be a rational function defined at each point of a closed interval $[a, b]$. Then for some $z \in (a, b)$,

$$f(b) = f(a) + f'(a)(b-a) + \frac{f''(z)}{2}(b-a)^2.$$

Proof: Set

$$g(x) = f(b) - f(x) - f'(x)(b-x) - C(b-x)^2,$$

where C is a constant. Then

$$\begin{aligned} g'(x) &= 0 - f'(x) - f''(x)(b-x) - f'(x)(-1) - 2C(b-x)(-1) \\ &= 0 - f'(x) - f''(x)(b-x) + f'(x) + 2C(b-x) \\ &= (b-x)[2C - f''(x)]. \end{aligned}$$

Also $g(b) = 0$. We choose the constant C so that $g(a) = 0$ as well which means we want:

$$0 = f(b) - f(a) - f'(a)(b-a) - C(b-a)^2;$$

i.e.

$$C = \frac{f(b) - f(a) - f'(a)(b-a)}{(b-a)^2}.$$

With this choice of C, Rolle's theorem tells us that for some $z \in (a, b)$, $g'(z) = 0$; i.e.

$$0 = g'(z) = (b-z)[2C - f''(z)].$$

Therefore dividing by $(b-z)$ we get $C = \frac{1}{2}f''(z)$; i.e.

$$\frac{f(b) - f(a) - f'(a)(b-a)}{(b-a)^2} = \frac{1}{2}f''(z)$$

and so

$$f(b) = f(a) + f'(a)(b-a) + \frac{f''(z)}{2}(b-a)^2.$$

$\square$