Contents

V Solving Polynomial Equations

Suppose $P(x)$ is a polynomial. We wish to consider the following problems.

Problem A. Find all the roots of $P(x)$ in a given interval $[a, b]$ to a given accuracy.

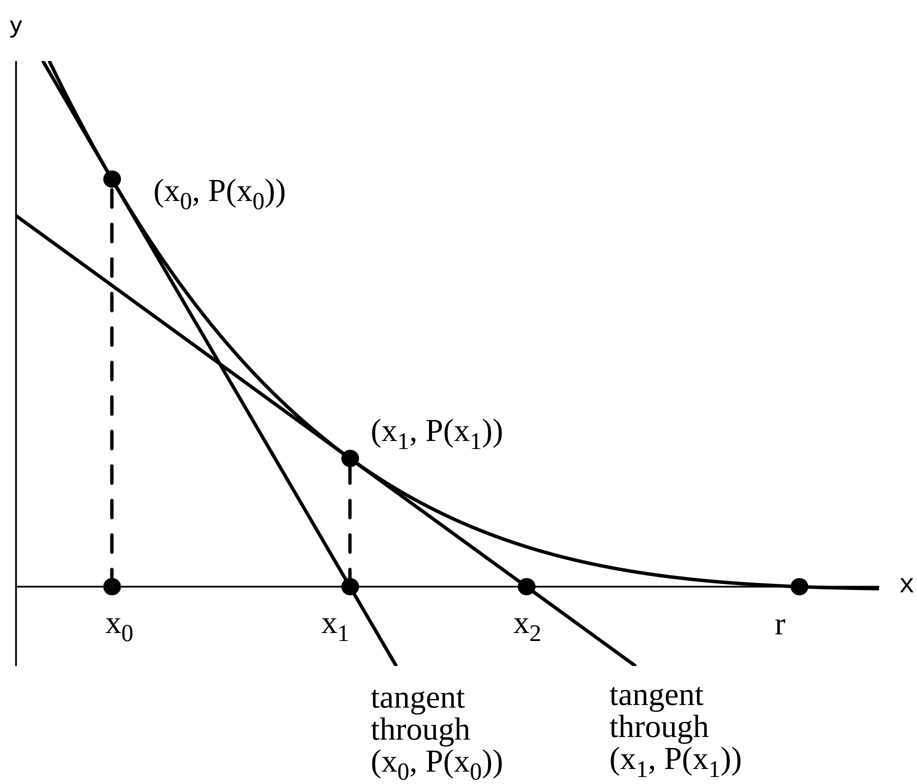
Problem B. Find all the roots of $P(x)$ to a given accuracy.

First we show how to solve problem A in a very special case, namely when $P'(x)$ has no roots in $[a, b]$ and $P(a)$ and $P(b)$ are non-zero and have opposite signs. Then we show how to solve the problem in general.

V.A Newton's Method in the Special Case

The algorithm we use in the special case is *Newton's method*.

The fundamental observation by Isaac Newton (1643–1727) was that if we know approximately where a root r of P is located, we may evaluate P at some point x_0 nearby. Usually, $P(x_0) \neq 0$, but from x_0 to the root r, the graph has to run from $P(x_0)$ to 0. As the tangent line to the graph of P at $(x_0, P(x_0))$ tends to point in the same direction as the graph, why not follow the tangent line until it intercepts the x–axis? The real stroke of genius was then to take x_1, the x–intercept of the tangent at $(x_0, P(x_0))$, as the new guess and to repeat the step, following now the tangent to the graph of P at $(x_1, P(x_1))$ — and to keep going!



Now we make things precise.

Theorem: Suppose that $P(x)$ is a polynomial such that

(i) $P'(x)$ has no root in $[a, b]$.

(ii) $P(a)$ and $P(b)$ are non-zero and have opposite signs, that is, $P(a)P(b) < 0$.

Then $P(x)$ has a exactly one root r in (a, b) and there is an algorithm to approximate r to any desired accuracy.

Proof: First we prove the assertion that $P(x)$ has a single root. Since $P(a)$ and $P(b)$ have opposite signs, the IVT tells us that $P(x)$ must vanish at some point $r \in (a, b)$; i.e. there is *at least* one root. On the other hand, Rolle's theorem says that two roots of $P(x)$ are separated by a root of $P'(x)$. Since $P'(x)$ has no roots in $[a, b]$, $P(x)$ cannot have two roots in $[a, b]$, it has *at most* one root.

The algorithm to approximate the root, r, consists of the following steps.

<u>Step 1</u> (Bounding derivatives): Find constants M_1 and M_2 so that

$$0 < M_1 \leq |P'(x)| \qquad \text{for all } x \in [a, b]$$

and

$$|P''(x)| \leq M_2 \qquad \text{for all } x \in [a, b].$$

(Important Note: $|P'(x)|$ has to be bounded *from below*, whereas $|P''(x)|$ has to be bounded *from above*!)

Let $K = \frac{M_2}{2M_1}$ and choose a constant $h > 0$ so that

$$Kh \leq 1.$$

<u>Step 2</u> (Shrinking the interval): Find inside $[a, b]$ an interval $[u, v]$ which contains the root, and such that $v - u \leq h$. This can be done as follows:

<u>Method 1</u> (Bisection):

- if $b - a \leq h$ set $u = a, v = b$. (The original interval is already narrow enough.)
- otherwise evaluate $P(x)$ at the midpoint of $[a, b]$.

  This divides $[a, b]$ into two equal subintervals and from the sign of $P(a)$, $P(b)$ and $P(\text{midpoint})$ you can tell which subinterval contains the root.
- This subinterval has length $\frac{1}{2}(b - a)$. If this length $\leq h$, declare the subinterval to be $[u, v]$. If not, choose its midpoint and continue the process until reaching an interval of length not exceeding h. (Unless you are very lucky and some midpoint = root!).

<u>Method 2</u> (Partition):

- Determine the factor $\triangle = \frac{b-a}{h}$ by which the length of the original interval exceeds the allowable length h.
- If $\triangle \leq 1$, the interval $[a, b]$ is clearly small enough and we take $u = a$, $v = b$.

- If $\triangle > 1$, find the largest integer n such that $n \leq \triangle < n+1$. Set
$$u_0 = a,\ u_1 = a+h, \ldots, u_j = a+jh, \ldots, u_n = a+nh,\ u_{n+1} = b.$$
Each interval $[u_{j-1}, u_j]$, for $j = 1, \ldots, n+1$, is of length at most h.
- Evaluate P at $u_0, u_1, \ldots$ until it changes sign, i.e. you find $P(u_{j-1})P(u_j) < 0$. Then $[u_{j-1}, u_j]$ contains the root r by IVT.
- Set $u = u_{j-1},\ v = u_j$.

(If some u_j is actually the root, consider yourself extremely lucky and stop right there.)

Note: Which method should one use?

If $\triangle$ is *small*, say at most 10, Method 2 is preferable, if $\triangle$ is *large*, bisection is faster.

Step 3: We now have our interval $[u, v]$ with $v - u \leq h$. Define a sequence of points $x_0, w_1, x_1, w_2, x_2, \ldots$ as follows:

- $x_0 = \frac{u+v}{2} \in [u, v]$, the midpoint of the interval.
- $w_{i+1} = x_i - \frac{P(x_i)}{P'(x_i)}$, this is the "magic formula".
- $x_{i+1} = \begin{cases} w_{i+1} & \text{if } w_{i+1} \in [u, v] \\ u & \text{if } w_{i+1} < u \\ v & \text{if } w_{i+1} > v \end{cases}$.

Step 4: We shall show by induction on $i \geq 0$ that the distance from the ith approximation x_i to the root r satisfies

$$(*) \qquad |r - x_i| \leq \frac{1}{K}\left(\frac{Kh}{2}\right)^{2^i} \leq \frac{1}{K}\left(\frac{1}{2}\right)^{2^i}.$$

It thus suffices to choose i such that $\frac{1}{K}\left(\frac{1}{2}\right)^{2^i} \leq \delta$, where δ is the required accuracy. The approximation x_i will then satisfy $|r - x_i| \leq \delta$.

Proof of Step 4:

(4.1): Let r be the root. By construction, $r \in [u, v]$ and each $x_i \in [u, v]$. Therefore by the EMVT

$$P(x) = P(x_i) + P'(x_i)(x - x_i) + \frac{P''(z)}{2}(x - x_i)^2$$

where z is some point between x_i and x. If we take $x = r$ then $P(r) = 0$ and this equation becomes

$$\text{(i)} \qquad 0 = P(x_i) + P'(x_i)(r - x_i) + \frac{P''(z)}{2}(r - x_i)^2.$$

Also, in Step 3, w_{i+1} was chosen such that

$$\text{(ii)} \qquad 0 = P(x_i) + P'(x_i)(w_{i+1} - x_i).$$

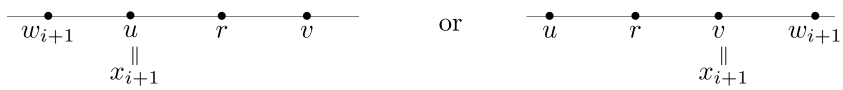
Subtract this from (i) to get

$$0 = P'(x_i)(r - w_{i+1}) + \frac{P''(z)}{2}(r - x_i)^2$$

i.e.

$$|r - w_{i+1}| = \left|\frac{P''(z)}{2P'(x_i)}\right|(r - x_i)^2.$$

If w_{i+1} is inside $[u, v]$ then $x_{i+1} = w_{i+1}$. If w_{i+1} is outside $[u, v]$:



then x_{i+1} is even closer to r than w_{i+1} and

$$|r - x_{i+1}| \leq |r - w_{i+1}|$$

therefore in any case

(iii)
$$|r - x_{i+1}| \leq \left|\frac{P''(z)}{2P'(x_i)}\right|(r - x_i)^2.$$

(4.2): Remember that $K = \frac{M_2}{2M_1}$; $Kh \leq 1$ by Step 1.

Since z is between r and x_i, both of which are in $[u, v]$, we have $z \in [u, v]$. Of course $x_i \in [u, v]$. Therefore by Step 1

$$|P''(z)| \leq M_2 \qquad \text{and} \qquad |P'(x_i)| \geq M_1,$$

and so $\left|\frac{P''(z)}{2P'(x_i)}\right| \leq \frac{M_2}{2M_1} = K$.

In view of 4.1, (iii) we get thus

$$|r - x_{i+1}| \leq K|r - x_i|^2.$$

(4.3): Now we are ready for the actual induction:

In fact, when $i = 0$, we have $r \in [u, v]$, and $[u, v]$ is of length at most h with midpoint x_0. Thus $|r - x_0| \leq h/2$; i.e.

$$|r - x_0| \leq \frac{1}{K} \cdot \frac{Kh}{2} = \frac{1}{K} \cdot \left(\frac{Kh}{2}\right)^{2^0}.$$

Suppose $(*)$ is true for some i. Then by (4.2),

$$\begin{aligned} |r - x_{i+1}| &\leq K|r - x_i|^2 \underset{\text{I.H.}}{\leq} K \cdot \left[\frac{1}{K} \cdot \left(\frac{Kh}{2}\right)^{2^i}\right]^2 \\ &= K \cdot \left(\frac{1}{K}\right)^2 \cdot \left[\left(\frac{Kh}{2}\right)^{2^i}\right]^2 \\ &= \frac{1}{K} \cdot \left(\frac{Kh}{2}\right)^{2^{i+1}}. \end{aligned}$$

Thus the first inequality in $(*)$ is proved by induction.

But $Kh \leq 1$, by Step 1, and so $\frac{Kh}{2} \leq \frac{1}{2}$ whence $\frac{1}{K} \cdot \left(\frac{Kh}{2}\right)^{2^i} \leq \frac{1}{K}\left(\frac{1}{2}\right)^{2^i}$. □

Example: Suppose $K = 1$. Then Step 4 says that

$$|r - x_i| < \left(\frac{1}{2}\right)^{2^i},$$

i.e.

$$\begin{aligned}
|r - x_0| &< \frac{1}{2} = 0.5 \\
|r - x_1| &< \left(\frac{1}{2}\right)^2 = \frac{1}{4} = 0.25 \\
|r - x_2| &< \left(\frac{1}{2}\right)^{2^2} = \left(\frac{1}{2}\right)^4 = \frac{1}{16} = 0.0625 \approx 6.3 \cdot 10^{-2} \\
|r - x_3| &< \frac{1}{256}, \\
|r - x_4| &< \left(\frac{1}{256}\right)^2 < \frac{1}{64000} = 0.00390625 \approx 3.9 \cdot 10^{-3}
\end{aligned}$$

and

$$\begin{aligned}
|r - x_5| &\leq \left(\frac{1}{64000}\right)^2 < \frac{1}{4000000000} \approx 1.525879 \cdot 10^{-5} \\
|r - x_6| &\leq 2.328306 \cdot 10^{-10}.
\end{aligned}$$

As a rule of thumb, you *double* the number of correct digits with each successive approximation.

Important Remarks.

- If we repeat Step 1 after Step 2 to find N_1 and N_2 such that

  $$0 < N_1 \leq |P'(x)| \qquad \text{and} \qquad |P''(x)| \leq N_2, \qquad \text{for all } x \in [u, v],$$

  and if we set $L = \frac{N_2}{2N_1}$ then L will usually be much smaller than K — and in particular $Lh \leq Kh \leq 1$, so that we don't have to repeat Step 2. We can use L instead of K in Step 4 and get a much better estimate of the error, so that we need less iterations to achieve the desired accuracy.

- In the actual application, one does Step 4 *before* Step 3 to know where to stop!

Example for Special Case.

Consider $P(x) = x^3 + 2x - 5$ on the interval $[1, 2]$. We have the special case because

i) $P'(x) = 3x^2 + 2 \neq 0$, and

ii) $P(1) = -2$ and $P(2) = 7$.

Estimate the root r of $P(x)$ in $[1, 2]$ to within $\pm 0.00005 = \pm 5 \cdot 10^{-5}$.

Step 1: $|P'(x)| = |3x^2 + 2| = 3x^2 + 2 \geq 3 \cdot 1^2 + 2 = 5$ on $[1, 2]$, so that we can take $M_1 = 5$.

$|P''(x)| = |6x| \leq 6 \cdot 2 = 12$ when $1 \leq x \leq 2$, so that we can take $M_2 = 12$.

With $K = \frac{M_2}{2M_1} = \frac{12}{2\cdot 5} = 1.2$, choose $h = \frac{1}{2}$ to get $Kh = 0.6 \le 1$.

<u>Step 2</u>: $P(1.5) = 1.375$, (evaluate at midpoint of $[1,2]$)

$P(1)$ and $P(1.5)$ have opposite signs, so $1 < r < 1.5$. Also, $1.5 - 1 = \frac{1}{2} \le h$.

Therefore we take $[u, v] = [1, 1.5]$.

<u>Repeat Step 1</u> on $[1, 1.5]$:

$$|P'(x)| = |3x^2 + 2| = 3x^2 + 2 \ge 5 \quad \text{— no better bound, } N_1 = M_1 = 5,$$

$$|P''(x)| = |6x| \le 6 \cdot 1.5 = 9 \quad \text{— better bound, } N_2 = 9 < M_2 = 12.$$

We can now take $K = \frac{N_2}{2\cdot N_1} = \frac{9}{2\cdot 5} = 0.9$ and we obtain $Kh = 0.9 \cdot 0.5 = 0.45$ (we did not change $h = \frac{1}{2}$!).

<u>Step 3</u>: $|r - x_i| \le \frac{1}{K}\left(\frac{Kh}{2}\right)^{2^i} = \frac{1}{0.9}(0.225)^{2^i}$. Try various values of i:

$i = 1$: $\frac{1}{0.9}(0.225)^2 = 0.05625$

$i = 2$: $\frac{1}{0.9}(0.225)^4 = 0.00285$

$i = 3$ $\frac{1}{0.9}(0.225)^8 = 0.0000073 < 0.00005$.

Thus we are assured that x_3 will be close enough to the root r.

<u>Step 4</u>: $x_0 = \frac{1+1.5}{2} = 1.25$ is the midpoint of the interval $[1, 1.5]$,

$$w_{i+1} = x_i - \frac{P(x_i)}{P'(x_i)} = x_i - \frac{x_i^3 + 2x_i - 5}{3x_i^2 + 2} = \frac{2x_i^3 + 5}{3x_i^2 + 2}$$

$w_1 = \frac{2(1.25)^3 + 5}{3(1.25)^2 + 2} = 1.331775 = x_1$, as w_1 is in $[1, 1.5] = [u, v]$,

$w_2 = 1.328276 = x_2$, as w_2 is in $[1, 1.5] = [u, v]$,

$w_3 = 1.328269 = x_3$, as w_3 is in $[1, 1.5] = [u, v]$.

Therefore $r = 1.328269 \pm 0.00005 \approx 1.3282$.

Comment: Without repeating Step 1, we obtain for $i = 3$ only the upper bound $\frac{5}{6}(0.3)^8 \approx 0.000055 > 0.00005$, as the "old" K was $1.2 = \frac{6}{5}$ instead of the "new" one 0.9. So we would have been forced to iterate at least one more time, as only for $i = 4$ we would get $\frac{5}{6}(0.3)^{16} \approx 3.6 \cdot 10^{-9} < 10^{-5}$.

V.B General Case

Here we wish to find all the roots of $P(x)$ in an interval $[a, b]$. The idea is to reduce to the special case by dividing $[a, b]$ into subintervals in which $P(x)$ either has no root or exactly one root.

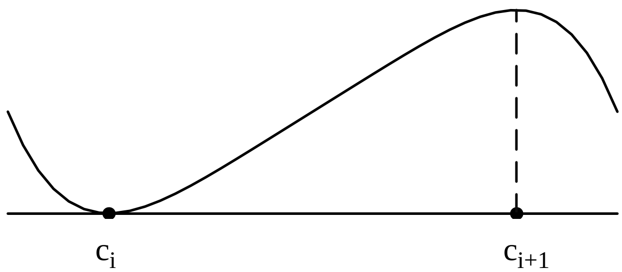
The procedure is broken into steps.

<u>Step 1</u>: Find the roots of $P'(x)$ in $[a, b]$ and label them $c_1, c_2, \cdots, c_k$. Call $a = c_0$ and $b = c_{k+1}$

Step 2: Evaluate $P(x)$ at $c_0, \ldots, c_{k+1}$.

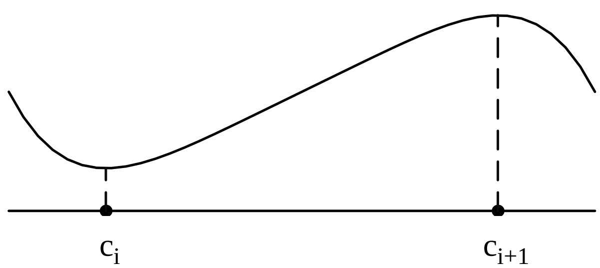
Step 3:

(i) If $P(c_i) = 0$ or $P(c_{i+1}) = 0$

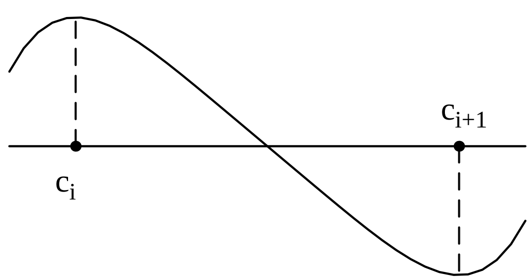


then $P(x)$ has *no root* in (c_i, c_{i+1}) — and we found one root of P at least!

(ii) If $P(c_i)P(c_{i+1}) > 0$, that is, $P(c_i) \neq 0$ and $P(c_{i+1}) \neq 0$ and they have the same sign, then $P(x)$ has *no root* in (c_i, c_{i+1}).



(iii) If $P(c_i)P(c_{i+1}) < 0$, that is, $P(c_i) \neq 0$ and $P(c_{i+1}) \neq 0$ and they have opposite signs, then $P(x)$ has *exactly one root* in (c_i, c_{i+1}).



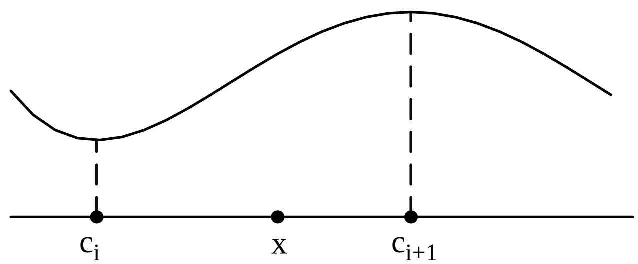
Justification of Step 3:

(i) Rolle's theorem tells us that between any two roots of $P(x)$ there must be a root of $P'(x)$; i.e. any two roots of $P(x)$ must be separated by one of the points c_j. Thus if c_i is a root of $P(x)$ $(P(c_i) = 0)$

then the next root of P must be to the right of c_{i+1}, and so P has no roots in (c_i, c_{i+1}).

Similarly if $P(c_{i+1}) = 0$ the next root of P to the left must be to the left of c_i;

(ii) Let x be any point in (c_i, c_{i+1}). We are given that $P(c_i)$ and $P(c_{i+1})$ have the same sign and we want to show that $P(x) \neq 0$.



Now by the CCS for P' and the MVT for P:

$$\textit{either } P' > 0 \text{ on } (c_i, c_{i+1}) \text{ and then} \qquad \begin{cases} P(x) - P(c_i) > 0 \\ P(c_{i+1}) - P(x) > 0 \end{cases}$$

$$\textit{or else } P' < 0 \text{ on } (c_i, c_{i+1}) \text{ and then} \qquad \begin{cases} P(x) - P(c_i) < 0 \\ P(c_{i+1}) - P(x) < 0. \end{cases}$$

If $P(x)$ were zero then we would have:

<u>In the first case</u>	<u>In the second case</u>
$P(c_{i+1}) > 0 > P(c_i)$	$P(c_{i+1}) < 0 < P(c_i)$

Since $P(c_{i+1})$ and $P(c_i)$ have the same sign this is impossible. Thus $P(x)$ cannot be zero.

(iii) Suppose $P(c_i)$ and $P(c_{i+1})$ have opposite signs. Then the IVT says that $P(x)$ must assume the value zero between c_i and c_{i+1}; i.e. $P(x)$ has at least one root in (c_i, c_{i+1}). But $P(x)$ cannot have two roots in (c_i, c_{i+1}) because they would have to be separated by a root of $P'(x)$. Thus $P(x)$ has exactly one root in (c_i, c_{i+1}).

Important Remark. At the end of Step 3 we can now count the number of roots of $P(x)$.

<u>Step 4</u>: In each of our subintervals $[c_j, c_{j+1}]$ with a root of $P(x)$ we know that $P(x)$ has opposite signs at the endpoints.

Choose points c and c' as shown above so that $P(c)$ and $P(c_j)$ have the same sign and $P(c')$ and $P(c_{j+1})$ have the same sign. Then

(i) $P'(x)$ has no root in $[c, c']$. (It is for this reason that we need to shrink $[c_j, c_{j+1}]$!)

(ii) $P(c)$ and $P(c')$ are non-zero and have opposite signs. We can thus apply the special case to the interval $[c, c']$ to locate the root of $P(x)$.

Solution to Problem B: Find all the roots of $P(x)$ in a given interval $[a, b]$ to a given accuracy.

We shall find a closed interval $[-R, R]$ such that all the roots of $P(x)$ are in $[-R, R]$. Using the procedure just described, we can then break up $[-R, R]$ into smaller intervals such that the solution of Problem A (the Special Case) applies in each of these smaller closed intervals.

To find the interval $[-R, R]$, write

$$P(x) = \sum_{i=0}^{n} a_i x^i, \quad a_n \neq 0,$$

and take

$$R = \max \left\{ 1, \sum_{i=0}^{n-1} \frac{|a_i|}{|a_n|} \right\}.$$

We now establish the

Theorem: All the roots of a polynomial $P(x) = \sum_{i=0}^{n} a_i x^i$ of degree n (i.e. $a_n \neq 0$) are in $[-R, R]$, where $R = \max\left\{1, \sum_{i=0}^{n-1} \left|\frac{a_i}{a_n}\right|\right\}$.

Proof: If λ is a root of $P(x)$ then

$$\sum_{i=0}^{n} a_i \lambda^i = 0 \Longleftrightarrow \lambda^n = -\sum_{i=0}^{n-1} \frac{a_i}{a_n} \lambda^i$$

and so

$$|\lambda|^n = \left| \sum_{i=0}^{n-1} \frac{a_i}{a_n} \lambda^i \right| \leq \sum_{i=0}^{n-1} \left| \frac{a_i}{a_n} \right| |\lambda|^i, \qquad \text{by the triangle inequality.} \tag{$*$}$$

Thus our claim is proved if we can show

$$|x|^n > \sum_{i=0}^{n-1} \left| \frac{a_i}{a_n} \right| |x|^i \qquad \text{whenever } |x| > R.$$

Now if $|x| > R$ then in particular $|x| > 1$, by the choice of R, and

$$1 = |x|^0 < |x|^1 < |x|^2 < \cdots < |x|^i < \cdots < |x|^{n-1}, \quad i = 1, 2, \ldots, n-1.$$

Thus

$$\begin{aligned}
\sum_{i=0}^{n-1} \left|\frac{a_i}{a_n}\right| \cdot |x|^i &\leq \left(\sum_{i=0}^{n-1} \left|\frac{a_i}{a_n}\right|\right) |x|^{n-1} && \text{, replacing each } |x|^i \text{ by } |x|^{n-1}, \\
&\leq R\ |x|^{n-1} && \text{, by definition of } R, \\
&< |x|\ |x|^{n-1} && \text{, by assumption on } x, \\
&= |x|^n.
\end{aligned}$$

Thus $|x|^n > \sum_{i=0}^{n-1} \left|\frac{a_i}{a_n}\right| |x|^i$ for $|x| > R$, and all roots must lie in $[-R, R]$ by $(*)$. □