# Contents

# VI Asymptotic Behaviour of Rational Functions

## VI.A Rational Functions as $x \to \pm\infty$

There exists a positive real constant $R$ such that all roots of $Q(x)$ are in $[-R, R]$ and so $f(x) = \frac{P(x)}{Q(x)}$ is defined whenever $x < -R$ or $R < x$, in other words, $x$ is *outside* the interval $[-R, R]$. We'll look at the behaviour of $f(x)$ when

i) $x < -R$ and $x \to -\infty$, or

ii) $R < x$ and $x \to \infty$.

PROPERTY 1: If deg $P <$ deg $Q$ then

$$f(x) \to 0 \qquad \text{as } x \to \pm\infty.$$

*Proof:* Write

$$P(x) = \sum_{i=0}^{n} a_i x^i; \quad a_n \neq 0 \quad \text{and} \quad Q(x) = \sum_{j=0}^{m} b_j x^j; \quad b_m \neq 0.$$

We have assumed $m > n$. Thus dividing by $x^m$ we find for $x \neq 0$

$$f(x) = \frac{(1/x^m)P(x)}{(1/x^m)Q(x)} = \frac{\sum_{i=0}^{n} a_i x^{i-m}}{\sum_{j=0}^{m} b_j x^{j-m}}.$$

*The numerator*: This has the form

$$\frac{a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0}{x^m} = \frac{a_n}{x^{m-n}} + \frac{a_{n-1}}{x^{m-n+1}} + \cdots + \frac{a_0}{x^m}.$$

Since $m > n$ each $m - i > 0$. Thus, as $|x| \to +\infty$, $|x^{m-i}| \to +\infty$ and $\frac{a_i}{x^{m-i}} \to 0$.

Therefore the numerator goes to 0 as $x \to \pm\infty$.

*The denominator*: This has the form

$$\begin{aligned} \frac{b_m x^m + \cdots + b_0}{x^m} &= b_m + \frac{b_{m-1}}{x} + \cdots + \frac{b_0}{x^m} \\ &\to b_m \quad \text{as } x \to \pm\infty. \end{aligned}$$

Therefore, the denominator goes to $b_m$ as $x \to \pm\infty$.

*The limit of $f$*:

$$f(x) = \frac{\text{numerator}}{\text{denominator}} \to \frac{0}{b_m} = 0 \quad \text{as } x \to \pm\infty$$

because $b_m \neq 0$. □

**Example**. For $f(x) = \frac{x^2+10^{10}}{x^3+x+1}$, deg(numerator) $-$ deg(denominator) $= 2 - 3 = -1 < 0$.

Therefore $f(x) \to 0$ as $x \to \pm\infty$. Indeed,

$f(x) = \frac{x^2+10^{10}}{x^3+x+1} = \frac{\frac{1}{x^3}\left(x^2+10^{10}\right)}{\frac{1}{x^3}(x^3+x+1)} = \frac{\frac{1}{x}+\frac{10^{10}}{x^3}}{1+\frac{1}{x^2}+\frac{1}{x^3}} \to \frac{0+0}{1+0+0} = 0$ as $x \to \pm\infty$.

PROPERTY 2: If $P(x) = \sum_{i=0}^{n} a_i x^i$, $a_n \neq 0$ and $Q(x) = \sum_{j=0}^{n} b_j x^j$, $b_n \neq 0$, have *same degree* then

$$f(x) \to \frac{a_n}{b_n} \neq 0 \qquad \text{as } x \to \pm\infty.$$

*Proof:* Using long division, we divide $Q(x)$ into $P(x)$, to get

$$P(x) = \frac{a_n}{b_n}Q(x) + R(x), \quad \text{with deg } R < n, \text{ or } R = 0.$$

Therefore

$$f(x) = \frac{P(x)}{Q(x)} = \frac{a_n}{b_n} + \frac{R(x)}{Q(x)}, \quad \text{with deg } R < \text{deg } Q, \text{ or } R = 0.$$

By Property 1, $\frac{R(x)}{Q(x)} \to 0$ as $x \to \pm\infty$. Therefore

$$f(x) \to \frac{a_n}{b_n} \quad \text{as } x \to \pm\infty.$$

□

**Example.** $f(x) = \frac{-x^2+1}{3x^2+10^{100}x}$, deg(numerator) − deg(denominator) $= 2 - 2 = 0$.

Therefore $f(x) \to \frac{a_n}{b_n} = \frac{-1}{3}$ as $x \to \pm\infty$.

PROPERTY 3: If deg $P >$ deg $Q$ then

$$|f(x)| \to \infty \quad \text{as } x \to \pm\infty.$$

*Proof:* $\frac{1}{f(x)} = \frac{Q(x)}{P(x)} \to 0$ as $x \to \pm\infty$, by Property 1. Therefore

$$\frac{1}{|f(x)|} \to 0 \qquad \text{as } x \to \pm\infty.$$

and so

$$|f(x)| \to \infty \qquad \text{as } x \to \pm\infty.$$

□

**Example.** $f(x) = \frac{x^6+x^2}{x^3-1}$, deg(numerator) − deg(denominator) $= 6 - 3 = 3 > 0$.

Therefore $|f(x)| \to \infty$ as $x \to \pm\infty$.

In the above properties we saw that

(1) deg $P <$ deg $Q \Rightarrow f(x) \to 0$ as $x \to \pm\infty$.

(2) deg $P =$ deg $Q \Rightarrow f(x) \to \frac{a_n}{b_n} \neq 0$ as $x \to \pm\infty$.

(3) deg $P >$ deg $Q \Rightarrow |f(x)| \to \infty$ as $x \to \pm\infty$.

Conversely, if we know $f(x) \to 0$ as $x \to \pm\infty$ then

$$\deg P \neq \deg Q, \qquad \text{because of (2)}$$

and

$$\deg P \not> \deg Q, \qquad \text{because of (3)}$$

Thus if $f(x) \to 0$ as $x \to \pm\infty$ we must have $\deg P < \deg Q$.

In the same way, if $f(x) \to c \neq 0$ then $\deg P = \deg Q$ and if $|f(x)| \to \infty$ then $\deg P > \deg Q$.

**Definition:** If $f(x) = \frac{P(x)}{Q(x)}$ is a rational function then the integer

$$r = \deg P - \deg Q$$

is called the *order* of $f$.

The number $r \in \mathbb{Z}$ can be any (*positive* or *negative*) integer; like the degree, it is only defined if $P$ is not the zero polynomial.

Thus the above properties can be summarized in the table

| | order | Possibilities for degrees | Behaviour at $\pm\infty$ |
|---|---|---|---|
| (1) | $r < 0$ | $\Leftrightarrow \deg P < \deg Q$ | $\Leftrightarrow f(x) \to 0 \quad$ as $x \to \pm\infty$. |
| (2) | $r = 0$ | $\Leftrightarrow \deg P = \deg Q$ | $\Leftrightarrow f(x) \to \frac{a_n}{b_n} \neq 0 \quad$ as $x \to \pm\infty$. |
| (3) | $r > 0$ | $\Leftrightarrow \deg P > \deg Q$ | $\Leftrightarrow \lvert f(x)\rvert \to \infty \quad$ as $x \to \pm\infty$. |

Property 4: If $f$ and $g$ are rational functions then

$$\begin{aligned} \text{order } (fg) &= \text{order } f + \text{order } g \\ \text{order } (f/g) &= \text{order } f - \text{order } g. \end{aligned}$$

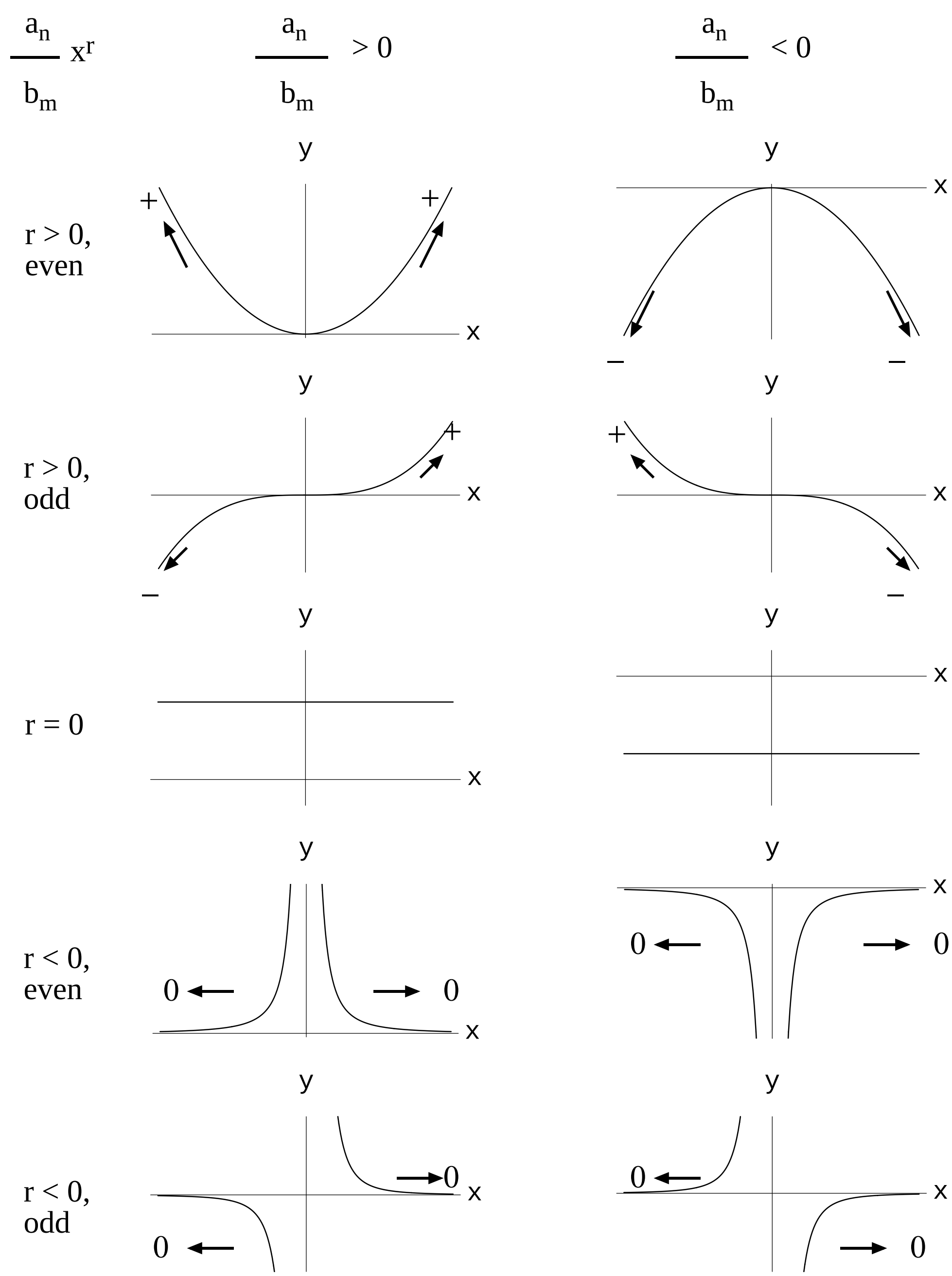
*Proof:* You check it.

We can refine the foregoing discussion a little bit.

Property 5: If $f(x) = \frac{P(x)}{Q(x)}$ is a rational function of order $r$, if $a_n$ is the leading coefficient of $P(x)$ and $b_m$ is the leading coefficient of $Q(x)$, then $r = n - m$ and

$$\frac{f(x)}{\frac{a_n}{b_m} x^r} \to 1 \quad \text{as } x \to \pm\infty.$$

In other words, if $|x|$ is very large, $f(x)$ and $\frac{a_n}{b_m}\, x^r$ "look alike". We can thus determine the behaviour of the graph of $f(x)$ for $x \to \pm\infty$ from that of $\frac{a_n}{b_m}\, x^r$:




*Proof:* By property 4, order $\left(\frac{f}{\frac{a_n}{b_m}x^r}\right) = \text{order}(f) - \text{order}\left(\frac{a_n}{b_m}x^r\right) = r - r = 0$. As

$$\frac{f}{\frac{a_n}{b_m}x^r} = \frac{\frac{1}{a_n}P(x)}{\frac{1}{b_m}Q(x)x^r} = \frac{x^n + \cdots}{x^{m+r} + \cdots}$$

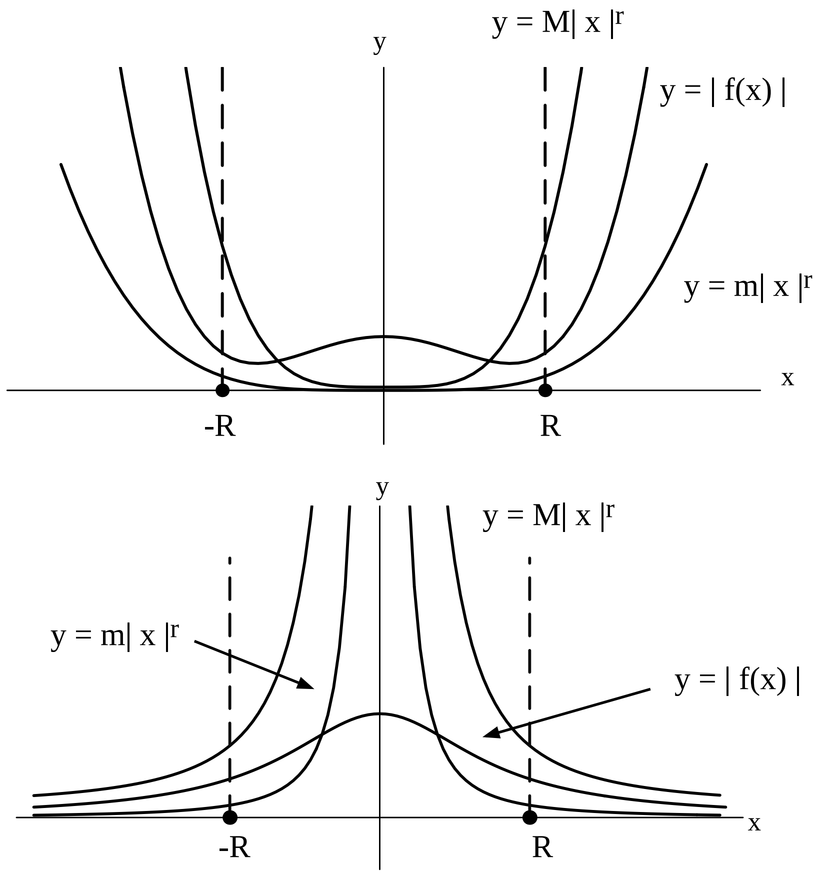
and $n = m + r$, property 2 yields the result. □

Quite often we need the following property, which is a slightly less precise version of property 5.

PROPERTY 6: Let $r$ be the order of a rational function $f(x)$. Then there are positive constants $R$, $m$ and $M$ such that

$$m|x|^r \le |f(x)| \le M|x|^r \quad \text{if} \quad |x| \ge R.$$

Before we see why this property is true, let us look at what it is saying in terms of the graph of $f$. We look at two pictures, corresponding to the cases $r > 0$ and $r < 0$.




Notice that

- Property 6 tells us about the behaviour of $|f(x)|$ *only* when $|x| \ge R$. It gives no information when $x \in (-R, R)$.
- Property 6 tells us that there are positive constants $m$, $M$, $R$. It does not say how to get them! If you look at the proof, which comes next, you may see how to get them.

*Proof of Property 6*: Consider the rational function $\frac{f(x)}{x^r}$. By Property 4,

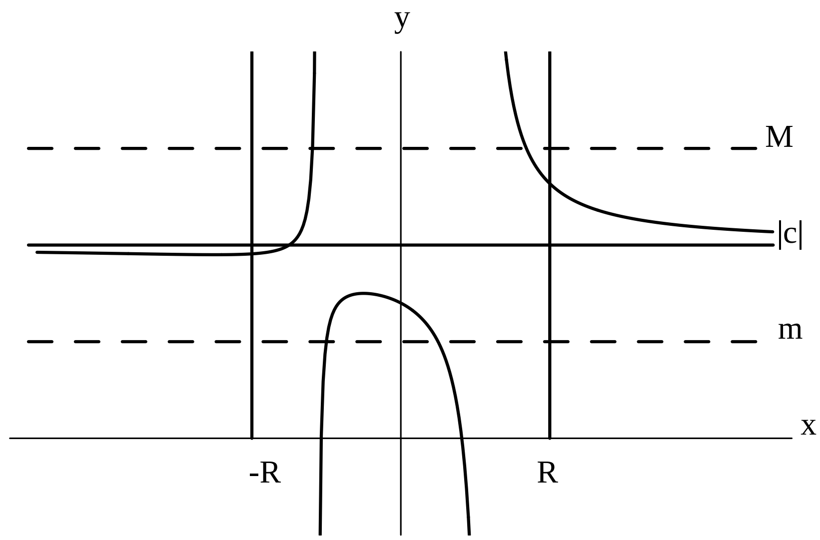
$$\text{order}\left(\frac{f(x)}{x^r}\right) = \text{order } f(x) - \text{order } (x^r) = r - r = 0.$$

By Property 2,

$$\frac{f(x)}{x^r} \to c \neq 0 \quad \text{as } x \to \pm\infty,$$

where $c$ is a non-zero real number — indeed, $c = \frac{a_n}{b_m}$, the quotient of the leading coefficients of numerator and denominator of $f$.

This means that for $|x|$ large enough, $\frac{f(x)}{x^r}$ is close to $c$ and so $\left|\frac{f(x)}{x^r}\right|$ is close to $|c|$. Graphically,




the graph of $\left|\frac{f(x)}{x^r}\right|$ gets close to the horizontal line $y = |c|$ as $x \to \pm\infty$.

Choose $M > |c|$, $m < |c|$ as marked on the $y$-axis. For large values of $|x|$, the graph of $\left|\frac{f(x)}{x^r}\right|$ will be trapped in the shade band, i.e. for some $R > 0$:

$$\text{if } |x| \geq R \quad \text{then } m \leq \left|\frac{f(x)}{x^r}\right| \leq M.$$

Multiply by $|x|^r$:

$$\text{if } |x| \geq R \text{ then } m|x|^r \leq |f(x)| \leq M|x|^r.$$

Thus Property 6 is verified. □

**Example**. If $f(x)$ is a rational function and $f'$ is not the zero function then

$$\frac{f'(x)}{f(x)} \text{ has order } r \leq -1$$

and so

$$\frac{f'(x)}{f(x)} \to 0 \text{ as } x \to \pm\infty.$$

*Proof:* Write $f = P/Q$. Then

$$\begin{aligned} \frac{f'}{f} &= \frac{P'Q - PQ'}{Q^2} \Big/ \frac{P}{Q} = \frac{P'Q - PQ'}{Q^2} \cdot \frac{Q}{P} \\ &= \frac{P'Q - PQ'}{PQ} \end{aligned}$$

and $\deg(P'Q - PQ') \leq \deg(PQ) - 1$, thus order $\left(\frac{f'}{f}\right) = \deg(P'Q - PQ') - \deg(PQ) \leq -1$.

## VI.B Rational Functions Near Zeros of the Denominator; Poles

Let $f(x) = \frac{P(x)}{Q(x)}$ be a rational function and suppose $a \notin \text{dom}(f)$ so that $Q(a) = 0$. Using the Factor Theorem we can write

$$\begin{aligned} P(x) &= (x-a)^m \cdot P_1(x) \quad \text{with } m \geq 0,\ P_1(a) \neq 0 \\ Q(x) &= (x-a)^n \cdot Q_1(x) \quad \text{with } n > 0,\ Q_1(a) \neq 0. \end{aligned}$$

So

$$f(x) = (x-a)^{m-n} \cdot \frac{P_1(x)}{Q_1(x)}$$

and $f_1(x) = \frac{P_1(x)}{Q_1(x)}$ is defined and nonzero at $a$.

If $m \geq n$, then $\lim\limits_{x \to a} f(x) = \lim\limits_{x \to a} (x-a)^{m-n} \cdot \lim\limits_{x \to a} f_1(x) = \begin{cases} 0 & \text{if } m > n \\ f_1(a) & \text{if } m = n. \end{cases}$

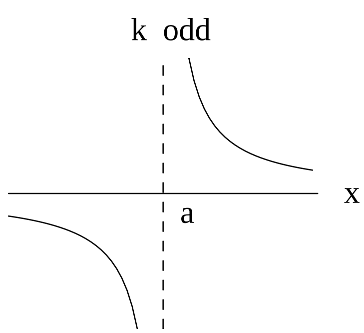
In the second case $(m = n)$, the graph of $f$ is the same as that of $f_1$ with $(a, f_1(a))$ removed.

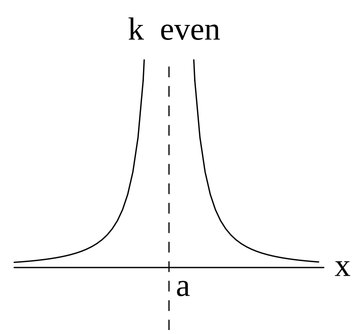
If $m < n$, set $k = n - m$ so that

$$f(x) = \frac{P_1(x)}{(x-a)^k Q_1(x)}.$$

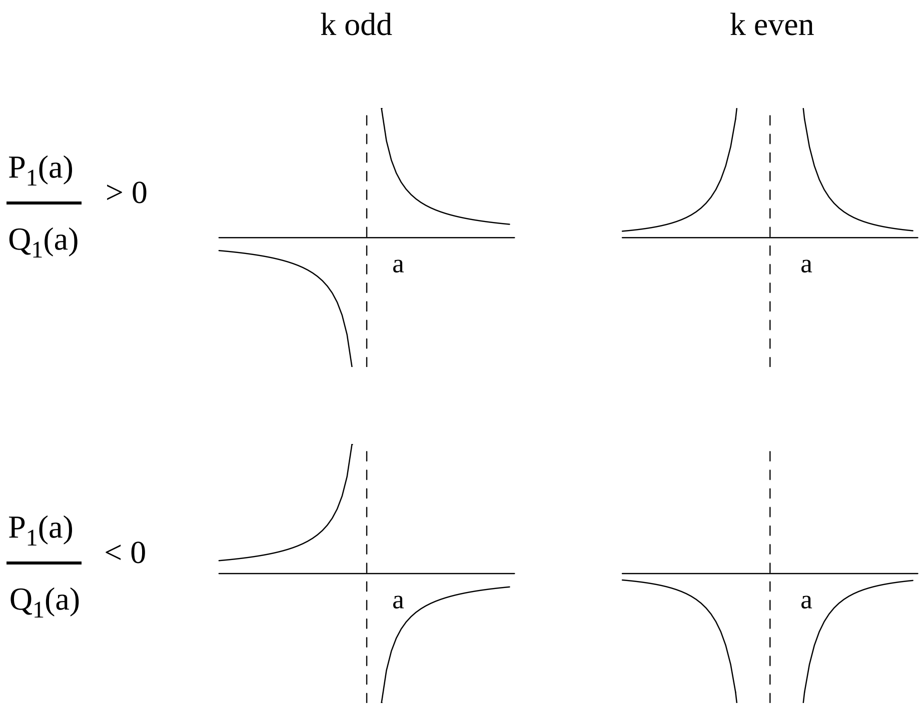
One has $\lim\limits_{x \to a} |f(x)| = \infty$, $f$ has a *pole of order $k$ at $a$.*

Look at the graphs of $\frac{1}{(x-a)^k}$:



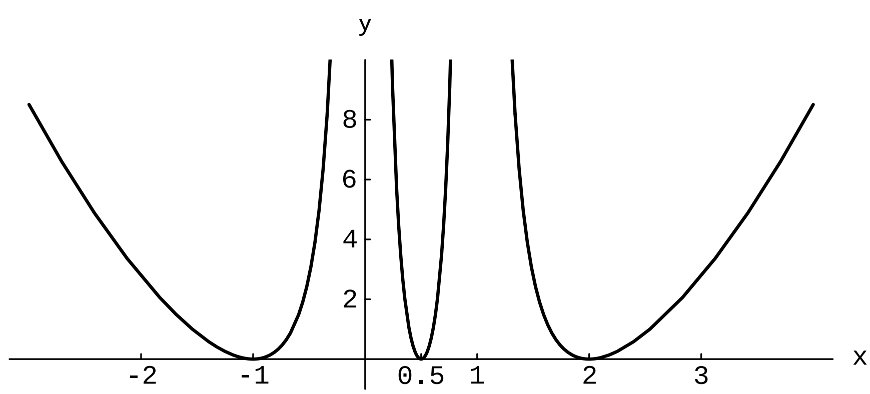




It follows that the graph of $f$ near $a$ looks like:




## VI.C Graph Versus Equation

Combining all the previous results, one can "guess" a rational function from its graph. We treat one example:




Find $f(x) = \frac{P(x)}{Q(x)}$ that matches the graph.

*order of f*: It is positive and even as $\lim\limits_{x\to\pm\infty} f(x) = +\infty$.

*zeros of f*: $-1, \frac{1}{2}, 2$ are the zeros, each at least of multiplicity two, $x$–axis being tangent.

*poles of f*: $f$ is not defined at 0, 1. It has poles of even order at those points.

*Conclusion*: $P(x) = (x+1)^2(x-\frac{1}{2})^2(x-2)^2 \cdot P_1(x);\ P_1(x) \neq 0,\ x \in \mathbb{R}$
$Q(x) = x^2(x-1)^2 Q_1(x);\ Q_1(x) \neq 0,\ x \in \mathbb{R}.$

Try: $f(x) = \alpha \frac{(x+1)^2(x-\frac{1}{2})^2(x-2)^2}{x^2(x-1)^2}.$

As order $f = 2$, we will have "correct" behaviour at $\pm\infty$. $\alpha$ must be positive as $f(x) \geq 0$ for all $x$.
*Answer*: $f(x) = \left(\frac{(x+1)(x-\frac{1}{2})(x-2)}{x(x-1)}\right)^2$ matches the graph.