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VII Taylor Approximation

VII.A L'Hospital's Rule

Assume f and g are functions defined near some $a \in \mathbb{R}$. If $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist *and* $\lim_{x\to a} g(x) \neq 0$, then one has

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)},$$

> "the limit of a quotient is the quotient of the limits — *provided* the limit of the denominator is *not zero*."

What about a limit like $\lim_{x\to 0} \frac{\sin x}{x}$? If one tries to take limits separately in numerator and denominator one ends up with an *indeterminate form* $\left[\frac{0}{0}\right]$ which means that both numerator and denominator tend to 0 with $x \to 0$. L'Hospital's rule sometimes allows one to determine limits of such indeterminate forms by differentiation.

In the example at hand it says

$$\lim_{x\to 0} \frac{\sin x}{x} = \frac{\lim_{x\to 0} (\sin x)'}{\lim_{x\to 0} (x)'} = \frac{\lim_{x\to 0} \cos x}{\lim_{x\to 0} 1} = \frac{\cos 0}{1} = \frac{1}{1} = 1,$$

so we obtain the limit by taking the quotient of the *limits of the derivatives* of numerator and denominator.

The precise formulation is the following.

Theorem: (l'Hospital's rule for $\left[\frac{0}{0}\right]$): Let f, g be functions and $a \in \mathbb{R}$. If

(i) f and g are differentiable in some interval $(a-h, a+h)$ with $h > 0$, and

(ii) $\lim_{x\to a} f(x) = 0$ *and* $\lim_{x\to a} g(x) = 0$, and

(iii) $\lim_{x\to a} \frac{f'(x)}{g'(x)}$ exists (allowing the limits $+\infty$ or $-\infty$),

then the limit $\lim_{x\to a} \frac{f(x)}{g(x)}$ exists as well and

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}. \qquad \square$$

The rule works also for limits of the *indeterminate form* $\left[\frac{\infty}{\infty}\right]$.

Theorem (l'Hospital for $\left[\frac{\infty}{\infty}\right]$): If f and g are functions that satisfy (i) and (iii) above together with

(ii′) $\lim_{x\to a} |f(x)| = \infty$ *and* $\lim_{x\to a} |g(x)| = \infty$,

then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}. \qquad \square$$

To repeat it: l'Hospital's rule works for those indeterminate forms where either *both* numerator and denominator tend to 0 or *both* tend to ∞.

Either form of l'Hospital's rule works for limits at infinity, $\lim\limits_{x\to+\infty} \frac{f(x)}{g(x)}$ or $\lim\limits_{x\to-\infty} \frac{f(x)}{g(x)}$, as well as for one-sided limits. In each case, condition (i) has to be adopted correspondingly. For example,

$$\lim_{x\to+\infty} \frac{f(x)}{g(x)} = \lim_{x\to+\infty} \frac{f'(x)}{g'(x)}$$

if (i) f and g are differentiable on some interval (b, ∞), i.e. "close to $+\infty$"; (ii) $\lim\limits_{x\to+\infty} f(x) = 0 = \lim\limits_{x\to+\infty} g(x)$ or $\lim\limits_{x\to+\infty} |f(x)| = \infty = \lim\limits_{x\to+\infty} |g(x)|$ and (iii) $\lim\limits_{x\to+\infty} \frac{f'(x)}{g'(x)}$ exists.

There are other indeterminate forms:

1. $\lim\limits_{x\to0^+} x \ln x$ is of the form $[0 \cdot \infty]$,
2. $\lim\limits_{x\to0^+} x^x$ is of the form $[0^0]$,
3. $\lim\limits_{x\to\infty} \left(1 - \frac{2}{x}\right)^{3x}$ is of the form $[1^\infty]$,
4. $\lim\limits_{x\to\infty} \left(e^{3x} - 5x\right)^{1/x}$ is of the form $[\infty^0]$,
5. $\lim\limits_{x\to0} \left(\frac{1}{x} - \frac{\sin x}{x^2}\right)$ is of the form $[\infty - \infty]$.

These indeterminate forms can usually be reduced to one of the forms $\left[\frac{0}{0}\right]$ or $\left[\frac{\infty}{\infty}\right]$:

Example 1. $\lim\limits_{x\to0^+} x \ln x$, (form $[0 \cdot \infty]$),

$= \lim\limits_{x\to0^+} \frac{\ln x}{\frac{1}{x}}$, $\left([0 \cdot \infty] \to \left[\frac{\infty}{\frac{1}{0}}\right] = \left[\frac{\infty}{\infty}\right]\right)$,

$= \lim\limits_{x\to0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}}$, $\left(\text{l'H. for } \left[\frac{\infty}{\infty}\right]; (\ln x)' = \frac{1}{x};\ \left(\frac{1}{x}\right)' = -\frac{1}{x^2}\right)$,

$= \lim\limits_{x\to0^+} (-x) = 0$, (this last limit clearly exists!).

Example 2.

$$\begin{aligned} & \lim_{x\to 0^+} x^x && ,\text{(form } [0^0]), \\ =\ & \lim_{x\to 0^+} e^{x\ln x} && ,\left([0^0] \to [e^{0\cdot\ln 0}] \to e^{[0\cdot\infty]}\right), \\ =\ & e^{\lim\limits_{x\to 0^+}(x\ln x)} && ,\left(\text{as } e^z \text{ is continuous and so } \lim_{x\to a} e^{f(x)} = e^{\lim_{x\to a} f(x)}\right), \\ =\ & e^0 = 1 && ,\text{(by Example 1.)}. \end{aligned}$$

Example 3.

$$\begin{aligned} & \lim_{x\to\infty} \left(1-\tfrac{2}{x}\right)^{3x} && ,\text{(form } [1^\infty]), \\ =\ & \lim_{x\to\infty} e^{3x\ln\left(1-\frac{2}{x}\right)} && ,\left([1^\infty] \to [e^{\infty\cdot\ln 1}] \to e^{[\infty\cdot 0]}\right), \\ =\ & e^{\lim\limits_{x\to\infty}(3x\ln(1-\frac{2}{x}))} && ,\text{(by continuity of } e^z \text{ again)}, \\ =\ & e^{-6}, \end{aligned}$$

as

$$\begin{aligned} & \lim_{x\to\infty} 3x\ln\left(1-\tfrac{2}{x}\right) = 3\lim_{x\to\infty} \frac{\ln\left(1-\frac{2}{x}\right)}{\frac{1}{x}} && ,\text{(form } [\tfrac{0}{0}]) \\ =\ & 3\lim_{z\to 0^+} \frac{\ln(1-2z)}{z} && ,\text{(set } \tfrac{1}{x} = z) \\ =\ & 3\lim_{z\to 0^+} \frac{-2/(1-2z)}{1} && ,\text{(l'H. for } [\tfrac{0}{0}]) \\ =\ & 3\lim_{z\to 0^+} \frac{-2}{1-2z} = -6. \end{aligned}$$

Example 4.

$$\begin{aligned} & \lim_{x\to\infty} \left(e^{3x}-5x\right)^{1/x} && ,\text{(form } [\infty^0]), \\ =\ & \lim_{x\to\infty} e^{\frac{1}{x}\ln(e^{3x}-5x)} && ,\left([\infty^0] \to [e^{0\ln\infty}] \to e^{[0\cdot\infty]}\right), \\ =\ & e^{\lim\limits_{x\to\infty}\left(\frac{\ln(e^{3x}-5x)}{x}\right)}, && \text{(continuity of } e^z), \\ =\ & e^3 \end{aligned}$$

as

$$\begin{aligned} \lim_{x\to\infty}\left(\frac{\ln(e^{3x}-5x)}{x}\right) & \overset{[\frac{\infty}{\infty}]}{=} \lim_{x\to\infty} \frac{3e^{3x}-5}{e^{3x}-5x} \\ & \overset{[\frac{\infty}{\infty}]}{=} \lim_{x\to\infty} \frac{3\cdot 3e^{3x}}{3e^{3x}-5} \\ & \overset{[\frac{\infty}{\infty}]}{=} \lim_{x\to\infty} \frac{3\cdot 9e^{3x}}{3\cdot 3e^{3x}} = \lim_{x\to\infty} \frac{27}{9} = 3. \end{aligned}$$

Example 5.

$$\begin{aligned}
&\lim_{x\to 0}\left(\tfrac{1}{x}-\tfrac{\sin x}{x^2}\right) &&\text{, (form } [\infty-\infty]),\\
=\ &\lim_{x\to 0}\tfrac{x-\sin x}{x^2} &&\text{, (common denominator } \to \left[\tfrac{0}{0}\right]),\\
=\ &\lim_{x\to 0}\tfrac{1-\cos x}{2x} &&\text{, (l'H. for } \left[\tfrac{0}{0}\right]),\\
=\ &\lim_{x\to 0}\tfrac{\sin x}{2} &&\text{, (l'H. for } \left[\tfrac{0}{0}\right] \text{ again)},\\
=\ &\tfrac{\sin 0}{2}=0.
\end{aligned}$$

VII.B Taylor Polynomials

If we want to approximate a function $f(x)$ by a constant near $x = a$ we use the constant approximation

$$A_0(x) \quad = \quad \lim_{x\to a} f(x) \quad (= f(a) \text{ if } f \text{ is continuous at } a).$$

If we want to approximate by a linear function near $x = a$ we use the linear approximation

$$A_1(x) \quad = \quad f(a) + f'(a)(x-a). \quad \text{(given by the tangent line to the graph at } x = a)$$

Note that the linear approximation is only defined if f is differentiable at $x = a$!

Clearly, $A_0(x)$ is a polynomial of degree (at most) 0 in x, whereas $A_1(x)$ is a polynomial of degree (at most) 1 in x.

These are approximating polynomials of degree 0 and degree 1. Now we introduce approximating polynomials of higher degree.

If f is a function defined near $a \in \mathbb{R}$, i.e. on an open interval $(a-\gamma, a+\gamma)$ with $\gamma > 0$, we want to find those polynomials $P_n(x)$ which approximate f best.

What do we mean by *best approximation* at a?

Definition: A polynomial $P_n(x)$ that is either the zero polynomial or of degree at most n is an n^{th} *order approximation of f at a* iff

$$\lim_{x\to a}\frac{f(x)-P_n(x)}{(x-a)^n} = 0.$$

If such a polynomial exists, it is also called the n^{th} *Taylor polynomial of f at a.*

Note: To avoid having to single out the zero polynomial all the time, we will make the convention that deg $0 = -\infty$, so that "deg $P \le n$" includes the zero polynomial.

Before we establish the *existence* of Taylor polynomials we show that they are *unique.* This relies upon the following simple Lemma on polynomials:

Lemma: Suppose $P(x)$ is a polynomial of degree at most n (including the possibility that $P(x) = 0$). Then for a real number $a \in \mathbb{R}$, $\lim_{x\to a}\frac{P(x)}{(x-a)^n} = 0$ iff $P(x) = 0$.

Proof. If $P(x) \neq 0$, we can write $P(x) = (x-a)^m Q(x)$, where $Q(x)$ is a polynomial with deg $Q \le n-m$ and $Q(a) \neq 0$. But then $\frac{P(x)}{(x-a)^n} = \frac{(x-a)^m Q(x)}{(x-a)^n} = \frac{Q(x)}{(x-a)^{n-m}}$ with $n-m \ge 0$, and

$$\lim_{x\to a}\frac{P(x)}{(x-a)^n} = \lim_{x\to a}\frac{Q(x)}{(x-a)^{n-m}} = \begin{cases} \text{does not exist} & \text{for } n-m>0\\ Q(a)\neq 0 & \text{for } n-m=0. \end{cases}$$

Thus we never get zero as a limit unless P is the zero polynomial! □

Now we can prove *uniqueness* of Taylor polynomials:

Proposition: If $n \geq 0$ is an integer and $f(x)$ is a function defined on $(a-h, a+h)$ for some $h > 0$, then there is *at most one* polynomial $P_n(x)$ such that

(1) $\deg P_n(x) \leq 0$ (including the possibility $P_n = 0!$)

(2) $\lim\limits_{x\to a} \frac{f(x)-P_n(x)}{(x-a)^n} = 0$.

Proof. Assume there were two polynomials $P_n(x)$, $Q_n(x)$ satisfying (1) and (2). Then, by (1), $\deg(P_n - Q_n) \leq n$, allowing again for $P_n = Q_n$! Also, by (2),

$$\begin{aligned} 0 = 0 - 0 &= \lim_{x\to a} \frac{f(x) - Q_n(x)}{(x-a)^n} - \lim_{x\to a} \frac{f(x) - P_n(x)}{(x-a)^n} \\ &= \lim_{x\to a} \frac{(f(x) - Q_n(x)) - (f(x) - P_n(x))}{(x-a)^n} \\ &= \lim_{x\to a} \frac{P_n(x) - Q_n(x)}{(x-a)^n}. \end{aligned}$$

Thus, by the Lemma, $P_n - Q_n$ is the zero polynomial, and $P_n(x) = Q_n(x)$ as desired. □

Note: If we want to explicitly mention f, we write P_n^f for the n^{th} Taylor polynomial of f.

What is the use of the Taylor polynomial — if it then exists?

$$\lim_{x\to a} \tfrac{f(x)-P_n(x)}{(x-a)^n} = 0 \qquad \text{implies that}$$

$$\text{for every } \varepsilon > 0, \qquad \left|\tfrac{f(x)-P_n(x)}{(x-a)^n}\right| < \varepsilon \qquad \text{on some interval } (a-\delta, a+\delta) \text{ with } \delta > 0.$$

Thus, on this interval

$$|f(x) - P_n(x)| < \varepsilon|x-a|^n < \varepsilon\delta^n$$

and $P_n(x)$ is a good approximation for $f(x)$ near a.

Now let us consider *existence*:

- for $n = 0$, we are looking for a constant $c \in \mathbb{R}$ (a polynomial of degree ≤ 0!) such that

$$\lim_{x\to a} \frac{f(x) - c}{(x-a)^0} = \lim_{x\to a} (f(x) - c) = 0.$$

  But this means $\lim\limits_{x\to a} f(x) = c$, in other words, the limit of $f(x)$ for $x \to a$ has to exist which happens iff f is *continuous* at a. (More precisely, f can be made continuous at a by declaring $f(a) = c$.)

- for $n = 1$, we are looking for a linear polynomial $P_1(x) = \alpha_0 + \alpha_1(x-a)$ such that

$$\lim_{x\to a} \frac{f(x) - \alpha_0 - \alpha_1(x-a)}{x-a} = 0,$$

  or, put differently,

$$\lim_{x\to a} \frac{f(x) - \alpha_0}{x-a} = \lim_{x\to a} \frac{\alpha_1(x-a)}{x-a} = \alpha_1,$$

  so the limit of $\frac{f(x)-\alpha_0}{x-a}$ has to exist for $x \to a$. But this means that

(a) $\lim_{x\to a}(f(x)-\alpha_0)=0$, i.e. f is *continuous* at a, and $f(a)=\alpha_0$ is the continuous extension of f at a,

(b) $\lim_{x\to a}\frac{f(x)-f(a)}{x-a}$ exists, i.e. f is *differentiable* at a, and

(c) $f'(a)=\lim_{x\to a}\frac{f(x)-f(a)}{x-a}=\alpha$.

In short, if the first Taylor polynomial exists, it is necessarily of the form

$$P_1(x)=f(a)+f'(a)(x-a),$$

the linear approximation to f at a.

To repeat it:

- $P_0(x)=f(a)$ is the *constant approximation* to f at a,
- $P_1(x)=f(a)+f'(a)(x-a)$ is the *linear approximation* to f at a, whose graph is the *tangent line* to the graph of f at $(a,f(a))$.

The uniqueness has the following useful consequences that show how to manipulate Taylor polynomials:

Corollary:

(i) If $P_n(x)=\alpha_0+\alpha_1(x-a)+\cdots+\alpha_n(x-a)^n$ is the n^{th} Taylor polynomial for f at a, then $P_{n-1}(x)=\alpha_0+\alpha_1(x-a)+\cdots+\alpha_{n-1}(x-a)^{n-1}$ is the $(n-1)^{\text{st}}$ Taylor polynomial for f at a.
("Dropping the term with x^n yields the previous Taylor polynomial").

(ii) If f is differentiable near a and P_n its n^{th} Taylor polynomial at a, whereas Q_{n-1} is the $(n-1)^{\text{st}}$ Taylor polynomial for f' at a, then

(a) $P_n(x)=f(a)+\int_a^x Q_{n-1}(t)dt$

(b) $P_n'(x)=Q_{n-1}(x)$

("Taylor polynomials of (anti-)derivatives are the (anti-)derivatives of the Taylor polynomials").

(iii) If f has n^{th} Taylor polynomial P_n and g has n^{th} Taylor polynomial Q_n, then

$$\begin{array}{lcl} \alpha f+\beta g & & \alpha P_n+\beta Q_n \\ \alpha f-\beta g & \text{has } n^{\text{th}} \text{ Taylor polynomial} & \alpha P_n-\beta Q_n \\ f\cdot g & & (P_n\cdot Q_n)_{\le n}, \end{array}$$

where $(P)_{\le n}$ means to take the polynomial and to drop all terms involving $(x-a)^m$ with $m>n$.

(iv) If $f(a)=0$ and $P_n(x)$ is the n^{th} Taylor polynomial of f at a, then $P_n(a)=0$ and $\frac{P_n(x)}{x-a}$ is the $(n-1)^{\text{st}}$ Taylor polynomial for $\frac{f(x)}{x-a}$.

Proof.

(i) We have to show $\lim_{x\to a}\frac{f(x)-P_{n-1}(x)}{(x-a)^{n-1}}=0$. But

$$\lim_{x\to a}\frac{f(x)-P_{n-1}(x)}{(x-a)^{n-1}}=\lim_{x\to a}\frac{f(x)-(P_n(x)-\alpha_n(x-a)^n)}{(x-a)^{n-1}}, \qquad \text{(by definition of } P_{n-1}(x))$$

$$=\lim_{x\to a}\left[\frac{(x-a)(f(x)-P_n(x))}{(x-a)(x-a)^{n-1}}+\alpha_n\frac{(x-a)^n}{(x-a)^{n-1}}\right]$$

$$=\lim_{x\to a}(x-a)\lim_{x\to a}\left(\frac{f(x)-P_n(x)}{(x-a)^n}\right)+\alpha_n\lim_{x\to a}(x-a) \qquad \text{(as all 3 limits exist)}$$

$$=0\cdot 0+0.$$

(ii) In case (a), $P_n(0) = f(a)$ and $P_n'(x) = Q_{n-1}(x)$ by the Fundamental Theorem of Calculus. Now use l'Hospital's rule:

$$\lim_{x\to a} \frac{f(x) - P_n(x)}{(x-a)^n} \overset{\text{(l'H)}}{=} \lim_{x\to a} \frac{f'(x) - Q_{n-1}(x)}{n(x-a)^{n-1}} = \frac{1}{n} \lim_{x\to a} \frac{f'(x) - Q_{n-1}(x)}{(x-a)^{n-1}} = 0$$

as Q_{n-1} is the $(n-1)^{\text{st}}$ Taylor polynomial for $f'(x)$ at a.

In case (b), observe that necessarily $P_n(a) = f_n(a)$ as otherwise $\lim_{x\to a} \frac{f(x)-P_n(x)}{(x-a)^n}$ could not possibly exist. Now

$$\begin{aligned} 0 &= \lim_{x\to a} \tfrac{f(x)-P_n(x)}{(x-a)^n} && \text{(assumption, as } P_n(x) \text{ is } n^{\text{th}} \text{ Taylor polynomial for } f \text{ at } a) \\ &= \lim_{x\to a} \tfrac{f'(x)-P_n'(x)}{n(x-a)^{n-1}} && \text{(by l'Hospital's rule)} \\ &= \tfrac{1}{n} \lim \tfrac{f'(x)-P_n'(x)}{(x-a)^{n-1}} \end{aligned}$$

and thus $\lim_{x\to a} \frac{f'(x)-P_n'(x)}{(x-a)^{n-1}} = 0$. As deg $P_n'(x) \le n-1$, $P_n'(x)$ is necessarily the $(n-1)^{\text{st}}$ Taylor polynomial for $f'(x)$ at a.

(iii) & (iv) You check it.

VII.C Taylor's Formula

Now we establish

Taylor's Theorem (without remainder): Assume $n \ge 0$ is an integer and $f(x)$ is n times differentiable at a. (In particular, $f(x)$ and its derivatives upto $f^{(n-1)}(x)$ are defined *near* a, whereas $f^{(n)}$ is defined at least *at* a.) Then

$$P_n(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

is the n^{th} Taylor polynomial of f at a.

Proof. Observe that for $0 \le i \le n$, $P_n^{(i)}(a) = f^{(i)}(a)$ and that $P_n^{(n)}(x) = f^{(n)}(a)$. Now use l'Hospital's rule:

$$\begin{aligned} \lim_{x\to a} \frac{f(x) - P_n(x)}{(x-a)^n} &= \lim_{x\to a} \frac{f'(x) - P_n'(x)}{n(x-a)^{n-1}} = \cdots = \quad \text{(the dots "...") are for induction!)} \\ &= \lim_{x\to a} \frac{f^{(n-1)}(x) - P_n^{(n-1)}(x)}{n(n-1)\cdots 2(x-a)} = \lim_{x\to a} \frac{f^{(n-1)}(x) - \left(f^{(n-1)}(a) + f^{(n)}(a)(x-a)\right)}{n!(x-a)} \\ &= \frac{1}{n!} \lim_{x\to a} \left(\frac{f^{(n-1)}(x) - f^{(n-1)}(a)}{x-a} \right) - \frac{1}{n!} f^{(n)}(a) = 0. \end{aligned}$$

Notice that l'Hospital's rule applies each time as

$$\lim_{x\to a} \left(f^{(i)}(x) - P_n^{(i)}(x) \right) = f^{(i)}(a) - P_n^{(i)}(a) = 0.$$

□

In general, $P_n(x)$ is that polynomial of degree at most n that fits the graph of f best among all those polynomials at $(a, f(a))$.

Example. Conversely, the graph of f tells us something about the Taylor polynomials.

If $P_2 = \alpha + \beta(x-a) + \gamma(x-a)^2$ is the second Taylor polynomial then $\alpha = f(a)$, $\beta = f'(a)$ and $\gamma = \frac{1}{2}f''(a)$. Thus

$\alpha > 0$ iff $f(a) > 0$, i.e. the graph is *above* the x–axis,

$\beta > 0$ iff $f'(a) > 0$, i.e. the graph *increases* at a,

$\gamma < 0$ iff $f''(a) < 0$, i.e. the graph is *concave down* at a, the approximating parabola open *downwards*.

How can we find the Taylor polynomials of a given function?

Examples.

By direct computation:

1. $f(x) = e^x$, $\quad f^{(i)}(x) = e^x \quad$ thus:
   $f^{(i)}(0) = 1$ for every $i \geq 0$ and the n^{th} Taylor polynomial at $a = 0$ is
   $$P_n(x) = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!}$$
2. $f(x) = \sin x,\ a = 0,$
   $f^{(4i)}(x) = \sin x \qquad f^{(4i)}(0) = 0$
   $f^{(4i+1)}(x) = \cos x \qquad f^{(4i+1)}(0) = 1$
   $f^{(4i+2)}(x) = -\sin x \qquad f^{(4i+2)}(0) = 0$
   $f^{(4i+3)}(x) = -\cos x \qquad f^{(4i+3)}(0) = -1$
   and
   $$P_n(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - + \cdots = \sum_{i=0}^{\lfloor n/2 \rfloor} (-1)^i \frac{x^{2i+1}}{(2i+1)!}.$$

By using the uniqueness and the above Corollary:

3. $f(x) = \frac{\sin x}{x}$, $a = 0$. From 2. and Corollary (iv) above:
   $$P^f_{n-1}(x) = \frac{P^{\sin x}_n(x)}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - + \cdots = \sum_{i=0}^{\lfloor n/2 \rfloor} (-1)^i \frac{x^{2i}}{(2i+1)!}.$$
   Note: You do not need to calculate $\left(\frac{\sin x}{x}\right)^{(i)} \Big|_{x=0}$ which is a mess at best!
4. $f(x) = Si(x) = \displaystyle\int_0^x \frac{\sin t}{t}\, dt$, the *sine-integral.* The n^{th} Taylor polynomial at $a = 0$ is
   $$P^f_n(x) = \int_0^x \underbrace{\left(1 - \frac{t^2}{3!} + \frac{t^4}{5!} - + \cdots\right)}_{\text{Taylor polynomial for } \frac{\sin t}{t} \text{ as just seen}} dt \quad \text{, by Corollary (ii)}$$
   $$= x - \frac{x^3}{3 \cdot 3!} + \frac{x^5}{5 \cdot 5!} - + \cdots = \sum_{i=0}^{\lfloor n/2 \rfloor} (-1)^i \frac{x^{2i+1}}{(2i+1)(2i+1)!}\ .$$
   So we can easily find the Taylor polynomials for $Si(x)$, although the definition of that function looks complicated at first!

Now we use the Taylor polynomials to find a limit:

5. Find $\lim\limits_{x\to 0} \frac{Si(x)-x}{x^3}$ if it exists.

   As $\lim\limits_{x\to 0} \frac{Si(x)-P_3(x)}{x^3} = 0$ by definition, we get

$$\begin{aligned}
\lim_{x\to 0} \frac{Si(x)-x}{x^3} &= \lim_{x\to 0} \left(\frac{Si(x)-x}{x^3} - \frac{Si(x)-P_3(x)}{x^3} \right) \\
&= \lim_{x\to 0} \frac{P_3(x)-x}{x^3} = \lim_{x\to 0} \frac{x - \frac{x^3}{3\cdot 3!} - x}{x^3} \\
&= -\frac{1}{3\cdot 3!} = -\frac{1}{18}.
\end{aligned}$$

   Thus the limit exists and equals $-\frac{1}{18}$.

VII.D The Remainder in Taylor's Formula

We think of $P_n(x)$ as an approximation to $f(x)$ and we write

$$f(x) = \underset{\text{approximation}}{P_n(x)} + \underset{\text{error}}{E_n(x)}$$

This error is sometimes called the *remainder* because it is what is left over after the approximation is taken away from $f(x)$.

Whenever we talk about "approximations", we need to control the error! For the Taylor polynomials there are two ways: through an integral or through a derivative.

Taylor's Theorem with remainder: Assume that f can be differentiated $(n+1)$ times on an interval $[b,c]$ containing a, and that $f^{(n+1)}$ is continuous there. Then the error term

$$E_n(x) = f(x) - P_n(x) = f(x) - \left(f(a) + \frac{f'(a)}{1!}(x-a) + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n \right)$$

satisfies

(1) (Integral formula) $E_n(x) = \displaystyle\int_a^x f^{(n+1)}(t)\frac{(x-t)^n}{n!}\,dt$ for all $x \in [b,d]$.

(2) (Lagrange's form) $E_n(x) = \frac{f^{(n+1)}(z)}{(n+1)!}(x-a)^{n+1}$ for some z between a and x.

Proof of the integral formula for $E_n(x)$: The proof is by induction on n.

We consider first the case $n=1$ and assume $f''(x)$ exists and is continuous in an interval $[b,c]$ containing a. Now fix $x \in [b,c]$. By the FTC,

$$f(x) - f(a) = \int_a^x f'(t)dt.$$

Therefore

$$f(x) = f(a) + \int_a^x f'(t)dt. \tag{1}$$

Now we use integration by parts on $\int_a^x f'(t)dt$:

$$u = f'(t) \qquad dv = dt$$

$$du = f''(t)dt \qquad v = t - x$$

Note that we are entitled to choose any v such that $dv = dt$; it turns out to be convenient to chose $t - x$ instead of t.

Integration by parts gives

$$\begin{aligned}
\int_a^x f'(t)dt &= \int_a^x u\,dv = uv\Big|_a^x - \int_a^x v\,du \\
&= f'(t)(t-x)\Big|_{t=a}^{t=x} - \int_a^x (t-x)f''(t)dt \\
&= -f'(a)(a-x) - \int_a^x f''(t)(t-x)dt \\
&= f'(a)(x-a) + \int_a^x f''(t)(x-t)dt.
\end{aligned}$$

Substituting this back in (1) gives

$$\begin{aligned}
f(x) &= f(a) + f'(a)(x-a) + \int_a^x f''(t)(x-t)dt \\
&= A_1(x) + E_1(x).
\end{aligned}$$

We have thus derived the

integral formula for $E_1(x)$:

$$E_1(x) = \int_a^x f''(t)(x-t)dt, \qquad x \in [b,c].$$

Integral formula for $E_n(x)$: Assume that f can be differentiated $n+1$ times in an interval $[b,c]$ containing a and that $f^{(n+1)}(x)$ is continuous. We shall derive the integral formula for $E_n(x)$:

$$E_n(x) = \int_a^x f^{(n+1)}(t)\frac{(x-t)^n}{n!}dt, \quad x \in [b,c].$$

Now we carry out the induction on n. We have already derived the formula for $E_1(x)$. We assume that for some $n \geq 2$

$$E_{n-1}(x) = \int_a^x f^{(n)}(t)\frac{(x-t)^{n-1}}{(n-1)!}dt.$$

Apply integration by parts with

$$u = f^{(n)}(t) \qquad dv = \frac{(x-t)^{n-1}}{(n-1)!}dt$$

$$du = f^{(n+1)}(t)dt \qquad v = -\frac{(x-t)^n}{n!}.$$

Thus

$$\begin{aligned}
E_{n-1}(x) &= \int_a^x u\,dv = uv\Big|_{t=a}^{t=x} - \int_{t=a}^{t=x} v\,du \\
&= -f^{(n)}(t)\frac{(x-t)^n}{n!}\Big|_{t=a}^{t=x} + \int_{t=a}^{t=x} f^{(n+1)}(t)\frac{(x-t)^n}{n!}dt,
\end{aligned}$$

and so

$$E_{n-1}(x) = \frac{f^{(n)}(a)}{n!}(x-a)^n + \int_a^x f^{(n+1)}(t)\frac{(x-t)^n}{n!}dt. \tag{2}$$

But now we may write

$$\begin{aligned} f(x) &= f_{n-1}(x) + E_{n-1}(x) \\ &= f(a) + f'(a)(x-a) + \cdots + \frac{f^{(n-1)}(a)}{(n-1)!}(x-a)^{n-1} + E_{n-1}(x). \end{aligned}$$

Substitute formula (2) for $E_{n-1}(x)$:

$$\begin{aligned} f(x) &= f(a) + \cdots + \frac{f^{(n-1)}(a)}{(n-1)!}(x-a)^{n-1} + \frac{f^{(n)}(a)}{n!}(x-a)^n + \int_a^x f^{(n+1)}(t)\frac{(x-t)^n}{n!}dt \\ &= f_n(x) + \int_a^x f^{(n+1)}(t)\frac{(x-t)^n}{n!}dt. \end{aligned}$$

In other words,

$$E_n(x) = f(x) - f_n(x) = \int_a^x f^{(n+1)}(t)\frac{(x-t)^n}{n!}dt, \quad x \in [b,c].$$

Proof of Lagrange's formula for $E_n(x)$: We assume $f(x)$ has $n+1$ derivatives in an interval $[b,c]$ containing a and that $f^{(n+1)}(x)$ is continuous in $[b,c]$. Fix $x > a$. By the EVT, $f^{(n+1)}$ takes on an absolute minimum and an absolute maximum on $[a,x]$. Let m_{n+1} be the absolute minimum of $f^{(n+1)}(t)$ on $[a,x]$ and let M_{n+1} be the absolute maximum of $f^{(n+1)}(t)$ on $[a,x]$. Then

$$m_{n+1} \le f^{(n+1)}(t) \le M_{n+1} \quad \text{for all } t \in [a,x]. \tag{3}$$

Also, by the IVT, $f^{(n+1)}(t)$ assumes every value between m_{n+1} and M_{n+1} as t ranges from a to x.

Now for $t \in [a,x]$, the difference $x-t$ is positive. Thus we can multiply (3) by $\frac{(x-t)^n}{n!}$ without changing the inequalities:

$$m_{n+1}\frac{(x-t)^n}{n!} \le f^{(n+1)}(t)\frac{(x-t)^n}{n!} \le M_{n+1}\frac{(x-t)^n}{n!}.$$

Integrating this we get

$$\int_a^x m_{n+1}\frac{(x-t)^n}{n!}dt \le \int_a^x f^{(n+1)}(t)\frac{(x-t)^n}{n!}dt \le \int_a^x M_{n+1}\frac{(x-t)^n}{n!}dt.$$

The middle term is exactly $E_n(x)$. The left and right hand terms can be evaluated explicitly; they are equal to $m_{n+1}\frac{(x-a)^{n+1}}{(n+1)!}$ and $M_{n+1}\frac{(x-a)^{n+1}}{(n+1)!}$. Thus

$$m_{n+1}\frac{(x-a)^{n+1}}{(n+1)!} \le E_n(x) \le M_{n+1}\frac{(x-a)^{n+1}}{(n+1)!}.$$

Therefore

$$m_{n+1} \le \frac{(n+1)!}{(x-a)^{n+1}}E_n(x) \le M_{n+1}.$$

In other words, $\frac{(n+1)!}{(x-a)^{n+1}}E_n(x)$ is a value between m_{n+1} and M_{n+1}, and so it is equal to $f^{(n+1)}(z)$ for some $z \in (a,x)$:

$$\frac{(n+1)!}{(x-a)^{n+1}}E_n(x) = f^{(n+1)}(z), \quad \text{some } z \in (a,x).$$

This also works for $x < a$ and so cross multiplying we get

Lagrange's formula for $E_n(x)$:

$$E_n(x) = \frac{(x-a)^{n+1}}{(n+1)!} f^{(n+1)}(z), \qquad \text{some } z \text{ between } a \text{ and } x. \qquad \square$$

In particular, if $K_{n+1} \geq |f^{(n+1)}(z)|$ for all $z \in [b, c]$ then

$$|E_n(x)| \leq \frac{|x-a|^{n+1}}{(n+1)!} K_{n+1}, \quad x \in [b, c].$$

Remark 1. Notice that if we are in a small interval about a, say $[a-h, a+h]$ with $h < 1$, then

$$|E_n(x)| \leq \frac{h^{n+1}}{(n+1)!} K_{n+1}, \quad x \in [a-h, a+h].$$

If all the K_k's are $\leq$ some constant C then

$$\text{for all } n, \quad |E_n(x)| \leq \frac{h^{n+1}}{(n+1)!} C, \quad x \in [a-h, a+h].$$

As n gets large, $h^{n+1} \to 0$, $\frac{1}{(n+1)!} \to 0$ and so $|E_n(x)| \to 0$ very quickly. Thus in this case $f_n(x)$ becomes a better and better approximation to $f(x)$ as $n \to \infty$.

Remark 2. When $n = 0$ Lagrange's formula says

$$f(x) - f(a) = (x-a) f'(z), \qquad \text{some } z \text{ between } a \text{ and } x,$$

and this is the MVT.

When $n = 1$ Lagrange's formula says

$$f(x) - f(a) - f'(a)(x-a) = \frac{(x-a)^2}{2} f''(z), \quad \text{some } z \text{ between } a \text{ and } x,$$

and this is the EMVT.

Thus the Lagrange formula is a "super" generalized mean value theorem.

VII.E Applications of Taylor's Formula

VII.E.1 l'Hospital's Rule Revisited

Proposition: Let f, g be functions defined near a. Assume that f has n^{th} Taylor polynomial P_n at a and that g has n^{th} Taylor polynomial Q_n at a.

If Q_n is not the zero polynomial, then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{P_n(x)}{Q_n(x)}.$$

Proof: Write the Taylor polynomial Q_n of g as

$$Q_n(x) = b_0 + b_1(x-a) + \cdots + b_n(x-a)^n.$$

As Q_n is not the zero polynomial, there is a smallest index $m \le n$ such that $b_m \ne 0$. Accordingly,

$$\frac{Q_n(x)}{(x-a)^m} = b_m + b_{m+1}(x-a) + \cdots + b_n(x-a)^{n-m}$$

is again a polynomial and $\lim_{x\to a} \frac{Q_n(x)}{(x-a)^m} = b_m \ne 0$. Now write

$$f(x) = P_n(x) + E_n^f(x) \quad \text{with} \quad \lim_{x\to a} \frac{E_n^f(x)}{(x-a)^n} = 0$$

and

$$g(x) = Q_n(x) + E_n^g(x) \quad \text{with} \quad \lim_{x\to a} \frac{E_n^g(x)}{(x-a)^n} = 0,$$

where E_n^f, E_n^g are thus the error terms in the n^{th} Taylor approximation. Then

$$\begin{aligned}
\lim_{x\to a} \frac{f(x)}{g(x)} &= \lim_{x\to a} \frac{P_n(x) + E_n^f(x)}{Q_n(x) + E_n^g(x)} \\
&= \lim_{x\to a} \frac{\frac{P_n(x)}{(x-a)^m} + \frac{E_n^f(x)}{(x-a)^m}}{\frac{Q_n(x)}{(x-a)^m} + \frac{E_n^g(x)}{(x-a)^m}} \quad \text{, with } m \le n \text{ as above,} \\
&= \frac{\lim_{x\to a}\left(\frac{P_n(x)}{(x-a)^m} + \frac{E_n^f(x)}{(x-a)^m}\right)}{\lim_{x\to a}\left(\frac{Q_n(x)}{(x-a)^m} + \frac{E_n^g(x)}{(x-a)^m}\right)},
\end{aligned}$$

as the limit in the denominator exists and is not zero — indeed, it equals b_m,

$$\begin{aligned}
&= \frac{\lim_{x\to a} \frac{P_n(x)}{(x-a)^m}}{\lim_{x\to a} \frac{Q_n(x)}{(x-a)^m}} \quad \text{, as } \lim_{x\to a} \frac{E_n^f(x)}{(x-a)^m} = \lim \frac{E_n^g(x)}{(x-a)^m} = 0, \\
&= \lim_{x\to a} \frac{P_n(x)}{Q_n(x)}.
\end{aligned}$$

□

Example. Find $\lim_{x\to 0} \frac{\cos 2x - \cos 5x}{3\sin^2 x}$.

$\sin x = x + E_2(x)$, as $P_2(x) = x$ is the 2^{nd} Taylor polynomial of $\sin x$,

$3(\sin x)^2 = 3x^2 + E_2(x)$, by Corollary (iii), from VII.B.

Thus $Q_2(x) = 3x^2$ is the second Taylor polynomial of the denominator and $\lim_{x\to 0} \frac{Q_2(x)}{x^2} = 3 \ne 0$. To apply the proposition we need hence the second Taylor polynomial of the numerator:

$\cos 2x = 1 - \frac{(2x)^2}{2!} + E_2^{\cos 2x}(x)$,

$\cos 5x = 1 - \frac{(5x)^2}{2!} + E_2^{\cos 5x}(x)$,

and so

$\cos 2x - \cos 5x = -\frac{(2x)^2}{2!} + \frac{(5x)^2}{2!} + E_2(x) = \frac{21}{2}x^2 + E_2(x)$, whence $P_2(x) = \frac{21}{2}x^2$ is the second Taylor polynomial of the numerator. Finally,

$$\lim_{x\to 0} \frac{\cos 2x - \cos 5x}{3\sin^2 x} = \lim_{x\to 0} \frac{P_2(x)}{Q_2(x)} = \lim_{x\to 0} \frac{\frac{21}{2}x^2}{3x^2} = \frac{21}{6} = \frac{7}{2}.$$

VII.E.2 Approximating in a Given Interval

Here we are given a function $f(x)$ in $[b, c]$ and a point $a \in [b, c]$, and we want to find an n such that the Taylor polynomial of f at a, $P_n(x)$, approximates $f(x)$ to a given accuracy in $[b, c]$.

Example. Find an n so that when $f(x) = e^x$ and $a = 0$, $P_n(x)$ approximates $f(x)$ in $[-1, 1]$ with an error of at most 10^{-5}.

We are trying to find n so that

$$|E_n(x)| \le 10^{-5}, \quad x \in [-1, 1].$$

We know by Lagrange's form of the remainder that for some z between 0 and x:

$$|E_n(x)| = \left| \frac{x^{n+1}}{(n+1)!} e^z \right| \le \frac{e}{(n+1)!}, \quad \text{if } x \in [-1, 1].$$

So we want to choose n so that $\frac{e}{(n+1)!} \le 10^{-5}$; i.e. so that

$$(n+1)! \ge 10^5 e.$$

Now $8! = 40320 \ge 10^5 e = 27828$. Thus we choose $n = 7$ and conclude that $P_7(x)$ will do:

$$e^x \sim 1 + x + \frac{x^2}{2!} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \frac{x^6}{720} + \frac{x^7}{5040}, \quad x \in [-1, 1],$$

with an error of at most 10^{-5}.

Example. Find a polynomial that approximates the sine-integral $Si(x) = \displaystyle\int_0^x \frac{\sin t}{t}\, dt$ with an error of at most 10^{-4} on $[-2, 2]$.

Solution: Start from $\sin x$: We try to find a suitable Taylor polynomial of $Si(x)$ at $a = 0$.

$$\begin{aligned} |\sin x - P_n(x)| &= \left| \int_0^x \sin^{(n+1)}(x) \frac{(x-t)^n}{n!}\, dt \right|, \quad \text{(by the integral form of the remainder)}, \\ &\le \left| \int_0^x \frac{(x-t)^n}{n!}\, dt \right| = \left| \frac{x^{n+1}}{(n+1)!} \right|. \quad (\text{as } |\sin^{(n+1)}(x)| \le 1), \end{aligned}$$

(This estimate works also with Lagrange's form). Thus

$$\left| \frac{\sin x}{x} - \underbrace{\frac{P_n(x)}{x}}_{(n-1)^{\text{st}} \text{ TP for } \frac{\sin x}{x}} \right| \le \left| \frac{x^n}{(n+1)!} \right|, \quad \left(\begin{array}{l} \text{clear for } x \ne 0; \text{ at } x = 0 \\ \text{use continuity of both sides!} \end{array} \right),$$

and so

$$\left|\int_0^x \frac{\sin t}{t}\,dt - \underbrace{Q_n(x)}_{n^{\text{th}}\text{ TP for }Si(x)}\right| = \left|\int_0^x \left(\frac{\sin t}{t} - \underbrace{P_{n-1}^{\frac{\sin x}{x}}(t)}_{=\frac{P_n(t)}{t}}\right) dt\right| \quad \text{, (by Corollary VII.B, (ii)),}$$

$$\leq \int_0^x \left|\frac{\sin t}{t} - \frac{P_n(t)}{t}\right| dt \qquad \text{(triangle inequality for Riemann sums)}$$

$$\leq \int_0^x \left|\frac{x^n}{(n+1)!}\right| dt = \left|\frac{x^{n+1}}{(n+1)(n+1)!}\right|.$$

This shows that on $[-2,2]$ we have

$$|Si(x) - Q_n(x)| \leq \left|\frac{x^{n+1}}{(n+1)(n+1)!}\right| \leq \frac{2^{n+1}}{(n+1)(n+1)!}.$$

Now $2^{10} = 1024$, $10! \approx 3.6 \times 10^6 \implies \frac{2^{10}}{10 \cdot 10!} \approx \frac{10^3}{3.6 \cdot 10^7} < 10^{-4}$, and so the 9^{th} Taylor polynomial of $Si(x)$ at 0 will do:

$$Si(x) \sim Q_9(x) = x - \frac{x^3}{3 \cdot 3!} + \frac{x^5}{5 \cdot 5!} - \frac{x^7}{7 \cdot 7} + \frac{x^9}{9 \cdot 9!}$$

with an error of at most 10^{-4} on $[-2,2]$.

If you want to use Taylor's formula directly, you are required to bound $Si^{(n)}(x)$ — a very unpleasant task!

VII.E.3 Binomial Theorem, Taylor's Version

The Taylor polynomials of $f^{(x)} = (1+x)^p$, p *any* real number, play an important role. One has

$$(1+x)^p = 1 + px + \frac{p(p-1)}{2}x^2 + \cdots + \frac{p(p-1)\cdots(p-n+1)}{n!}x^n + E_n(x).$$

The n^{th} derivative of $f(x)$ is

$$f^{(n)}(x) = [(1+x)^p]^{(n)} = p(p-1)\cdots(p-n+1)(1+x)^{p-n},$$

as one verifies by induction on n; see also Example 8 in I.C. Thus

$$f^{(n)}(0) = p(p-1)\cdots(p-n+1)$$

and the result follows from Taylor's theorem.

Example. $\lim_{x\to 0} \frac{\cos x - \sqrt{1-x^2}}{x^4} = \lim_{x\to 0} \frac{\left(1-\frac{1}{2}x^2+\frac{1}{4!}x^4+E_4\right)-\left(1-\frac{1}{2}x^2-\frac{1}{8}x^4+E_4\right)}{x^4} = \lim_{x\to 0} \frac{\frac{1}{4!}x^4+\frac{1}{8}x^4}{x^4} = \frac{1}{6}.$