Contents

VIII Complex Numbers

Definition: A *complex number* is an expression of the form $x + iy$ where x and y are real numbers. If

$$z = x + iy$$

is a complex number, then we call x and y the *real* and *imaginary parts* of z and write

$$x = \text{Re } z \qquad \text{and} \qquad y = \text{Im } Z.$$

Thus both Re z and Im z are real numbers.

In the special case that $y = \text{Im } z$ is zero we write simply

$$z = x + i0 = x. \qquad (z \text{ is called real})$$

If instead $x = \text{Re } z$ is zero we write

$$z = 0 + iy = iy. \qquad (z \text{ is called purely imaginary})$$

In particular we write

$$\begin{aligned} 0 &= 0 + i0 \\ i &= 0 + i1. \end{aligned}$$

The set of all complex numbers is denoted by $\mathbb{C}$. It contains the set $\mathbb{R}$ of all real numbers:

$$\mathbb{R} = \{z \in \mathbb{C} \mid \text{Im } z = 0\}.$$

VIII.A Algebraic Operations

We *add* (or *subtract*) complex numbers by adding (or subtracting) real and imaginary parts:

$$(x + iy) \pm (x_1 + iy_1) = (x \pm x_1) + i(y \pm y_1).$$

We *multiply* complex numbers by the rule:

$$(x + iy)(x_1 + iy_1) = (xx_1 - yy_1) + i(xy_1 + x_1y). \tag{1}$$

This may seem a little bizarre at first, and quite hard to remember. However it is easy to remember if you follow two simple rules. Firstly a special case of (1):

$$i^2 = (0 + i1)(0 + i1) = (0 - 1) + i(0 + 0) = -1;$$

i.e.

$$i^2 = -1. \tag{2}$$

Secondly to get (1) from (2) must expand out normally:

$$\begin{aligned} (x + iy)(x_1 + iy_1) &= xx_1 + x(iy_1) + (iy)x_1 + (iy)(iy_1) \\ &= xx_1 + i(xy_1) + iyx_1 + i^2yy_1 \\ &= (xx_1 - yy_1) + i(xy_1 + yx_1). \end{aligned}$$

Multiplication of complex numbers has the same elementary properties as it does for real numbers:

$$z_1(z_2z_3) = (z_1z_2)z_3 \qquad (\textit{associativity})$$

$$zw = wz \qquad (\text{commutativity})$$

$$z(w_1 + w_2) = zw_1 + zw_2$$

$$1 \cdot z = z$$

$$0 \cdot z = 0.$$

Even better, if $z \neq 0$ we can divide by z.

In fact, suppose $z = x + iy$ is nonzero. Then at least one of x, y is nonzero and since these are *real* numbers, $x^2 + y^2 > 0$. Thus we can form the complex number (here z' is just another complex number; not the derivative)

$$z' = \frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2}.$$

Now multiply out

$$\begin{aligned} z'z &= \left(\frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2}\right)(x+iy) \\ &= \left(\frac{x^2}{x^2+y^2} + \frac{y^2}{x^2+y^2}\right) + i\left(\frac{xy}{x^2+y^2} - \frac{yx}{x^2+y^2}\right) \\ &= 1. \end{aligned}$$

Thus $z' = 1/z$:

$$\frac{1}{z} = \frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2}.$$

More generally we can divide any w by z

$$\frac{w}{z} = w \cdot \frac{1}{z} = w \cdot \left(\frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2}\right).$$

Example. $\frac{1}{2+3i} = \frac{1}{2+3i} \cdot \frac{2-3i}{2-3i} = \frac{2-3i}{4+9} = \frac{2}{13} - \frac{3}{13}i.$

VIII.B Advantages and Disadvantages

Expanding the reals to the complexes has advantages and disadvantages:

<u>On the positive side</u>. We can now factor more polynomials into linear factors: for instance

$$\begin{aligned} t^3 + t^2 + t + 1 &= (t+1)(t^2+1) \\ &= (t+1)(t+i)(t-i). \end{aligned}$$

<u>On the negative side</u>. In the complex numbers the sum of squares can now be zero: for instance

$$1^2 + (i)^2 = 1 - 1 = 0.$$

(This is because we can take square roots of negative numbers.) Actually this is very useful (not really negative at all), but until you get used to it you can easily fool yourself into making a mistake (and that is a negative).

VIII.C Complex Conjugate and Absolute Value

Let $z = x + iy$ be a complex number. Its *complex conjugate*, $\bar{z}$, is the complex number

$$\bar{z} = x - iy.$$

Thus

$$\bar{z} = z \Leftrightarrow z \text{ is real}, \qquad \bar{z} = -z \Leftrightarrow z \text{ is purely imaginary.}$$

Moreover

$$z\bar{z} = (x + iy)(x - iy) = x^2 - (iy)^2 = x^2 + y^2$$

is a nonnegative *real* number. Thus we define

$$|z| = \sqrt{z\bar{z}} = \sqrt{x^2 + y^2};$$

it is called the *absolute value* of z. Note also that

$$\begin{aligned} z^{-1} &= \frac{\bar{z}}{|z|^2}, \\ \overline{zw} &= \bar{z}\,\bar{w}, \end{aligned}$$

and

$$|zw| = |z|\,|w|.$$

VIII.D Exponentials

Earlier in this course we defined the *exponential* e^x, of a real number x. Now we wish to define the exponential of a complex number $z = x + iy$:

Definition: $e^z = (e^x \cos y) + i(e^x \sin y) = e^x(\cos y + i \sin y)$.

Again, at first sight this seems a strange definition. We choose it because of the three properties below:

Property 1. If z is real ($z = x$) then

$$e^z = e^{x+i.0} = e^x \cos 0 + ie^x \sin 0 = e^x;$$

i.e. we have not changed the definition for real numbers.

Property 2. $e^z e^w = e^{z+w}$.

Proof:

$$\begin{aligned} e^{x+iy}e^{a+ib} &= e^x(\cos y + i \sin y)e^a(\cos b + i \sin b) \\ &= e^x e^a[(\cos y \cos b - \sin y \sin b) + i(\cos y \sin b + \sin y \cos b)] \\ &= e^{x+a}[\cos(y + b) + i \sin(y + b)] \\ &= e^{(x+a)+i(y+b)}. \end{aligned}$$

Property 3. $\overline{e^z} = e^{\bar{z}}$.

Proof:

$$\begin{aligned}\overline{e^z} &= \overline{e^{x+iy}} = \overline{e^x(\cos y + i\sin y)} \\ &= e^x\cos y - ie^x\sin y \\ &= e^x(\cos(-y) + i\sin(-y)) \\ &= e^{x+i(-y)} = e^{x-iy} \\ &= e^{\bar{z}}.\end{aligned}$$

For purely imaginary number $i\theta$ we have

$$\cos\theta + i\sin\theta = e^{i\theta} \qquad \cos\theta - i\sin\theta = e^{-i\theta}.$$

Adding these equations and dividing by 2 we get

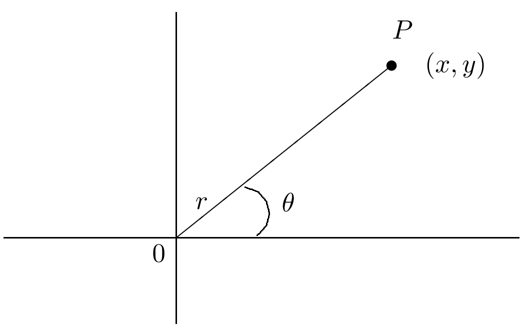
$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}.$$

Subtracting and dividing by $2i$ we get

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}.$$

VIII.E Polar Coordinates and the Complex Plane

A complex number $x + iy$ can be thought of as just a pair (x, y) of real numbers; i.e. it can be represented by a point P in the (x, y) plane. When we describe the point by the pair of real numbers x, y we call them the *Cartesian coordinates* of P.



However, the point P can also be described by another pair of numbers r, θ where

$$r = \text{length of } OP \qquad \theta = \text{ angle } OP \text{ makes with positive } x\text{-axis.}$$

Notice that θ is measured in radians, and is chosen so $0 \le \theta < 2\pi$.

We can express r and θ in terms of x and y because of the relationships

$$r = \sqrt{x^2 + y^2} \qquad \tan\theta = \frac{y}{x}.$$

It is even easier to express x and y in terms of r:

$$x = r\cos\theta \qquad y = r\sin\theta.$$

Now suppose $z = x + iy$ is a complex number. Then we can write

$$z = r\cos\theta + ir\sin\theta = r(\cos\theta + i\sin\theta) = re^{i\theta},$$

and so we arrive at the *polar form* of z:

$$z = re^{i\theta}.$$

Here $r = \sqrt{x^2 + y^2}$ is the absolute value of z, and so the complex numbers of length 1 (corresponding to the circle of radius 1 in the (x, y) plane) are just the numbers of the form $z = e^{i\theta}$, $0 \le \theta < 2\pi$.

VIII.F Complex Valued Functions of a Real Variable

A *complex-valued* function of a *real variable* is a rule $z(t)$ which assigns to each real number t in its domain a complex number $z(t)$. Thus

domain consists of *real* numbers   range consists of *complex* numbers.

We can write

$$z(t) = x(t) + iy(t);$$

then $x(t)$ and $y(t)$ are ordinary real-valued functions of a real variable t and

$$x(t) = \text{Re } z(t), \qquad y(t) = \text{Im } z(t).$$

The function $z(t)$ is *continuous* or *differentiable* if both $x(t)$ and $y(t)$ are; also

$$\begin{aligned} \lim_{t\to t_0} z(t) &= \left[\lim_{t\to t_0} x(t)\right] + i\left[\lim_{t\to t_0} y(t)\right] \\ z'(t) &= x'(t) + iy'(t). \end{aligned}$$

and

$$\int_{t_0}^{t} z(u)du = \left[\int_{t_0}^{t} x(u)du\right] + i\left[\int_{t_0}^{t} y(u)du\right].$$

VIII.G Product, Power and Quotient Rules

$$\begin{aligned} [z(t)\cdot w(t)]' &= z'(t)\cdot w(t) + z(t)\cdot w'(t) \\ ([z(t)]^n)' &= nz'(t)z(t)^{n-1} \\ [z(t)/(w)]' &= \frac{z'(t)w(t) - z(t)w'(t)}{w(t)^2}. \end{aligned}$$

Exponentials. Suppose $z(t)$ is a complex valued function of t. Then so is $e^{z(t)}$. Let us calculate the derivative:

$$\begin{aligned} \left[e^{z(t)}\right]' &= \left[e^{x(t)}\cos y(t) + ie^{x(t)}\sin y(t)\right]' \\ &= \left[e^{x(t)}\cos y(t)\right]' + i\left[e^{x(t)}\sin y(t)\right]' \\ &= \left[x'(t)e^{x(t)}\cos y(t) - y'(t)e^{x(t)}\sin y(t)\right] \end{aligned}$$

$$
\begin{aligned}
& +i\left[x'(t)e^{x(t)}\sin y(t) + y'(t)e^{x(t)}\cos y(t)\right] \\
= \;& [x'(t) + iy'(t)] \cdot \left[e^{x(t)}\cos y(t) + ie^{x(t)}\sin y(t)\right] \\
= \;& z'(t)e^{z(t)}.
\end{aligned}
$$

Thus

$$\left(e^{z(t)}\right)' = z'(t)e^{z(t)}. \qquad \text{(complex multiplication)}$$

Examples.

(i) $(t^2 - it^3)' = 2t - 3it^2$.

(ii)

$$
\begin{aligned}
\left(\frac{1}{t^2 + i(t+1)}\right)' &= \frac{-(2t+i)}{(t^2 + i(t+1))^2} \\
&= \frac{-2t - i}{t^4 - (t+1)^2 + 2it^2(t+1)} \\
&= \frac{(-2t - i)[t^4 - (t+1)^2 - 2it^2(t+1)]}{[t^4 - (t+1)^2]^2 + 4t^4(t+1)^2} \\
&= \frac{[-2t(t^4 - (t+1)^2 - 2t^2(t+1)] + i[-t^4 + (t+1)^2 + 4t^3(t+1)]}{[t^4 - (t+1)^2]^2 + 4t^4(t+1)^2}.
\end{aligned}
$$

(iii) $(e^{iat})' = iae^{iat}$

$(e^{it^2})' = 2ite^{it^2}$.