Contents

IX Gold Problems

1. i) State the Remainder Theorem.
   ii) Prove that $x^n - c^n$ is divisible by $x - c$ and compute the quotient where n is a positive integer.
   iii) Under what conditions is $x^n + c^n$ divisible by $x + c$.

2. Use the remainder theorem to show that

   a) $p(x) = x^4 - 8x^2 + 4x + 3$ is divisible by $x + 3$.
   b) $p(x) = 2x^4 - 7x^3 - 2x^2 + 13x + 6$ is divisible by $x^2 - 5x + 6$.

3. For the following functions $f(x) = \frac{P(x)}{D(x)}$ express $f(x)$ in the form $f(x) = Q(x) + \frac{R(x)}{D(x)}$ and compute $f'(x)$ in a form free of negative exponents.

   i) $f(x) = \frac{x^2}{x^2-1}$.
   ii) $f(x) = \frac{x^2+1}{x-2}$.
   iii) $f(x) = \frac{x^3}{x-2}$.
   iv) $f(x) = \frac{x^3-b}{x^3+a}$.
   v) $f(x) = \frac{x^3-3}{x^2+x+1}$.
   vi) $f(x) = \frac{x+1}{x^2+x+1}$.

IX.A Roots of Polynomials and Rational Functions

4. Show that $x^3 + ax + b = 0$ has exactly one real root if $a \geq 0$ and at most one real root between $-\frac{1}{3}\sqrt{3\,|a|}$ and $\frac{1}{3}\sqrt{3\,|a|}$ if $a < 0$.

5. Show that $ax^3 + bx^2 + cx + d = 0$ has exactly one real root if $b^2 - 3ac < 0$.

6. Let $P(x) = x^3 + cx + d$. Prove that if $c > 0$ there is exactly one real zero of P.

7. Let f be a polynomial. Suppose that f has a double root at a and at b. Show that $f'(x)$ has at least three roots.

8. Let $P(x)$ be a polynomial. Show that $(x - a)^2$ divides $P(x)$ if and only if $P(a) = 0$ and $P'(a) = 0$.

9. i) By evaluating $P(x) = x^3 + x + 1$ at -1 and at 0 show that $P(x)$ has at least one root.
   ii) Show that if $a < b$ then $\frac{P(b)-P(a)}{b-a} \geq 1$.
   iii) How many roots does $P(x)$ have?

10. Does there exist a function f such that $f(0) = -1$, $f(2) = 4$ and $f'(x) \leq 2$ for all x.

11. Show that $x^5 + 10x + 3 = 0$ has exactly one real root.

12. Show that $x^7 + 5x^3 + x - 6 = 0$ has exactly one real root.

13. Show that $x^5 - 6x + c = 0$ has at most one distinct root in the interval $[-1, 1]$.

14. Show that $x^4 + 4x + c = 0$ has at most two distinct real roots.

15. Give three examples, not all the same type, of polynomials that have no real roots.

16. Prove $f_\alpha(x) = x^3 - 3x + \alpha$ never has two roots in $[0, 1]$.

17. Show that $6x^4 - 7x + 1 = 0$ does not have more than two distinct real roots.

18. i) Show that $6x^5 + 13x + 1 = 0$ has exactly one real root.
    ii) Show that $x^3 + 9x^2 + 33x - 8 = 0$ has exactly one real root.

19. i) Show that a polynomial with only two non-zero coefficients has at most three distinct roots.
    ii) Show that a polynomial with only three non-zero coefficients has at most five distinct roots.
    iii) Find an example of a polynomial with only three non-zero coefficients and five roots.
    iv) Can you find examples of polynomials with only two non-zero coefficients that have
        a) no roots?
        b) one root?
        c) two roots?
        d) three roots?

20. Prove by induction that a polynomial with only k non-zero coefficients has at most $2k - 1$ distinct roots.

21. For how many points $z \in [0, 10]$ does the MVT hold for $f(x) = \frac{x}{1+x^2}$?

22. Let $P(x) = x^3 - 7x^2 + 7$. Show that $P(x)$ has at most one distinct root in $[1, 2]$.

23. Let $P(x) = x^5 + x - 1$. Show that $P(x)$ has at most one distinct root.

24. For each of the following, use IVT and/or Rolles Theorem to determine the number of roots
    i) $f(x) = x^{12} + 2x - 1$.
    ii) $f(x) = 3x^8 + 7x^4 + 2$.
    iii) $f(x) = \frac{1}{7}x^7 + x^4 + 6x + 1$.

    For each $f(x)$ above and each root r of $f(x)$ find an interval $[a, b]$ containing r.

25. i) Use the IVT to prove the following result.
    Let f be a rational function defined in the interval $[a, b]$ and suppose that $f(x) \neq c$ for every $x \in [a, b]$.
    If $f(a) < c$ then $f(b) < c$.
    If $f(a) > c$ then $f(b) > c$.
    ii) Use part i) to show that if f, a rational function, is defined on $[0, 3]$, f has no roots on this interval, and that $f(0) = 1$. Then $f(x) > 0$ for all x in $[0, 3]$.

26. The function $f(x) = \frac{1}{x-1}$ never takes the value zero yet $f(0) = -1$ is negative and $f(2) = 1$ is positive. Why isn't this a counterexample to the IVT.

27. Suppose $f(x)$ is a rational function all of whose values are integers. Show that f is constant.

28. Suppose $f(x)$ and $g(x)$ are rational functions whose values are the same at infinitely many points. Show that $f(x) = g(x)$ whenever both are defined.

29. Suppose $f(x)$ is a rational function defined at each point of $[a, b]$. If all values of $f(x)$ are rational numbers show that f is constant on $[a, b]$.

30. Let

$$f(x) = \begin{cases} c, & \text{if } x \in [a, b] \\ (x-a)^2 + c, & \text{if } x < a \\ -(x-b)^2 + c, & \text{if } x > b \end{cases} .$$

    Prove f is not a rational function.

    Hint: Consider $g(x) = f(x) - c$.

31. Suppose $f(x)$ is a rational function defined at each point of (a, b) and never zero there. Show that f has constant sign in (a, b).

32. For each of the following rational functions $f(x)$:

    i) Give the domain.

    ii) Mark on the real line, $\mathbb{R}$, the points where $f(x)$ is zero or undefined.

    iii) The points of ii) divide $\mathbb{R}$ into open intervals. Use problem 31 to conclude that $f(x)$ has constant sign in each interval. Find the sign and mark it ($+$ or $-$) above each interval.

    iv) Find the first and second derivatives of f

    a) $f(x) = \frac{1}{1-x^2} - 1$

    b) $f(x) = \frac{9x-3}{x^3-9x} + 1$

    c) $f(x) = \frac{x}{x^3-3x} - 1$

    d) $f(x) = \frac{4}{3} + \frac{7}{3(3x-1)}$

    e) $f(x) = 7 - \frac{1}{1+x^2}$

    f) $f(x) = \frac{x^2}{x^2-3}$.

33. To which of the following does Rolles' Theorem apply? Why?

    i) $f(x) = \frac{(x+3)(x-2)}{x^2+5x+4}$; $[-3, 2]$

    ii) $f(x) = \frac{(x+3)(x-2)}{x^2+5x+4}$; $[0, 3]$

    iii) $f(x) = \frac{(x+3)(x-2)}{x^2+9x+20}$; $[-3, 2]$

    when it is satisfied find a point z such that the conclusion of Rolles' Theorem holds.

34. i) Determine whether or not Rolles' theorem applies to each of the following functions on the given interval:

    a) $f(x) = \frac{(x+2)(x-1)}{x^2+4x+3}$; $[-2, 1]$

    b) $f(x) = \frac{(x+2)(x-1)}{x^2+4x+3}$; $[1, 2]$

    c) $f(x) = \frac{(x+2)(x-1)}{x^2+8x+16}$; $[-2, 1]$

    d) $f(x) = \frac{(x+2)(x-1)}{x^2-3x-10}$; $[-2, 1]$

    ii) When it is satisfied find a point z such that the conclusion of Rolles' theorem holds.

    iii) For which of the functions/intervals above does the MVT hold?

35. Suppose that

    i) f satisfies the hypotheses of the MVT on the interval $[a, b]$.

    ii) $m \leq f'(c) \leq M$ if $a < c < b$.

    Prove: $f(a) + m(x-a) \leq f(x) \leq f(a) + M(x-a)$.

36. Prove that if

$$\frac{a_0}{1} + \frac{a_1}{2} + \cdots + \frac{a_4}{5} = 0$$

then $a_0 + a_1 x + a_2 x^2 + \cdots + a_4 x^4 = 0$ for some x in $[0, 1]$.

Hint:

$$\left(\sum_{i=0}^{4} a_i \frac{x^{i+1}}{i+1} \right)' = \sum_{i=0}^{4} a_i x^i.$$

37. Suppose f is a rational function with $[a,b] \subset \text{domain}\,(f)$, and assume $f'(x)$ is constant in $[a,b]$.

    i) Show that there are constants m and c such that

    $$f(x) = mx + c, \qquad x \in [a,b].$$

    ii) Show that $f(x) = mx + c$, all $x \in \text{domain}\,(f)$.

38. Suppose $f(x)$ is a rational function defined at each point of $[a,b]$ and assume that $f'(x)$ is constant in $[a,b]$. Show that $f(x)$ is a straight line. (What is its slope? What is its y-intercept?)

39. To which of the following does the mean value theorem (MVT) apply? Why?

    i) $f(x) = \frac{x^2}{x^2-1}$; $[0,3]$

    ii) $f(x) = \frac{x^2}{x^2-1}$; $[2,5]$

    iii) $f(x) = \frac{x^3-5}{x^3-16x}$; $[-3,-1]$

    iv) $f(x) = \frac{x^3-5}{x^3-16x}$; $[-3,1]$

    v) $f(x) = \frac{(x+2)(x-1)}{x}$; $[1,2]$

    vi) $f(x) = \frac{x+1}{x^2+1}$; $[0,1]$

    vii) $f(x) = \frac{(x+2)(x-1)}{x^2-3x-10}$; $[-2,1]$

    when it is satisfied find a point z such that the conclusion of MVT holds.

40. Redo the assigned parts of question 39 with EMVT instead of MVT.

41. To which of the following does the mean value theorem apply? Why?

    i) $f(x) = \frac{1}{1-x^2} - 1$; $[0,2]$

    ii) $f(x) = \frac{1}{1-x^2} - 1$; $[2,4]$

    iii) $f(x) = \frac{9x-3}{x^3-9x} + 1$; $[-2,-1]$

    iv) $f(x) = \frac{9x-3}{x^3-9x} + 1$; $[-1,0]$

    State the MVT explicitly for each example above where it does apply: (e.g. when $f(x) = x^3$ and the interval is $[3,4]$ the MVT states that $4^3 - 3^3 = f'(z)(4-3) = 3z^2(4-3)$, some $z \in (3,4)$; i.e. $z = \sqrt{\frac{37}{3}}$.)

42. For each of the following rational function $f(x)$

    i) Give the domain.

    ii) Mark on the real line, $\mathbb{R}$, the points where $f(x)$ is zero or undefined.

    iii) The points of ii) divide $\mathbb{R}$ into open intervals. Conclude that $f(x)$ has constant sign in each interval. Find the sign and mark ($+$ or $-$) above each interval.

    iv) Find the first and second derivations of f:

    a) $f(x) = \frac{1}{x^2-2} + 2$
    b) $f(x) = \frac{9}{2} + \frac{5}{2(2x-1)}$
    c) $f(x) = \frac{x^2+2x-2}{x^3-x^2-2x} + 1$

43. To which of the following does the mean value theorem for rational functions apply? Explain your answer.

    i) $f(x) = \frac{1}{x^2-2} + 2 \quad ; [1, 2]$

    ii) $f(x) = \frac{1}{x^2-2} + 2 \quad ; [2, 3]$

    iii) $f(x) = \frac{x^2+2x-2}{x^3-x^2-2x} + 1 \quad ; [1, 3]$

    iv) $f(x) = \frac{x^2+2x-2}{x^3-x^2-2x} + 1 \quad ; [-4, -2]$.

44. i) Determine whether or not Rolles' Theorem applies to each of the following functions on the given interval:

    a) $f(x) = \frac{(x-2)(x+3)}{x^2-x-20} \quad ; \ [-3, 2]$
    b) $f(x) = \frac{(x-2)(x+3)}{x^2-x-20} \quad ; \ [-2, 3]$
    c) $f(x) = \frac{(x-2)(x+3)}{x^2+6x+8} \quad ; \ [-3, 2]$

    ii) When it is satisfied find a point z such that the conclusion of Rolles' Theorem for rational functions holds.

    iii) For which of the function/intervals above does the MVT (for rational functions) hold. (Explain.)

45. To which of the following does the MVT (for rational functions) apply? Why?

    i) $f(x) = \frac{x-1}{x^2-3} \quad ; [1, 3]$

    ii) $f(x) = \frac{x-1}{x^2-3} \quad ; [2, 3]$

    iii) $f(x) = \frac{x^2-1}{x^3-16x} \quad ; [-2, -1]$

    iv) $f(x) = \frac{x^2-1}{x^3-16x} \quad ; [-2, 1]$.

    When it is satisfied find a point z such that the conclusion of MVT (for rational functions) holds.

46. Redo question 45 with EMVT instead of MVT. (This time it is *not* required to find the z.)

47. To which of the following does Rolles' Theorem for rational functions apply? Why?

i) $f(x) = \frac{(x+1)(x-3)}{x^2-2x-8}$; $[-3, 1]$

ii) $f(x) = \frac{(x+1)(x-3)}{x^2-2x-8}$; $[-1, 3]$

iii) $f(x) = \frac{(x+1)(x-3)}{x^2-4x}$; $[-1, 3]$

when it is satisfied find a point z such that the conclusion of Rolles' Theorem holds.

48. To which of the following does the Mean Value Theorem for rational functions apply? Why?

i) $f(x) = \frac{x}{x^2-2}$; $[0, 3]$

ii) $f(x) = \frac{x}{x^2-2}$; $[2, 4]$

iii) $f(x) + \frac{x+2}{x^3-25x}$; $[-4, -1]$

iv) $f(x) = \frac{x+2}{x^3-25x}$; $[-4, 1]$

when it is satisfied find a point z such that the conclusion of MVT (for rational functions) holds.

49. Redo question 48 with EMVT instead of MVT. (This time it is *not* required to find the z.)

IX.B Increasing and Decreasing Functions

50. For the following functions, find where the function is increasing and where it is decreasing.

i) $g(x) = |3x^2 + 4x|$ ii) $g(x) = |x^2 + 1|$

iii) $g(x) = |x^2 - 2|$ iv) $g(x) = |x^2 + x - 4|$

v) $g(x) = |x^2 + x|$ vi) $g(x) = |x^2 + x + 1|$

vii) $g(x) = |x^2 - 4|$ viii) $g(x) = |x^2 - x + 2|$

51. For each of the following functions, determine the intervals where g is increasing and decreasing.

i) $g(z) = |3z^2 - 4z|$ ii) $g(z) = |z^2 - 2|$

iii) $g(z) = |z^2 + z + \sqrt{3}|$ iv) $g(z) = |z^2 + 9z + 20|$

v) $g(z) = |-z^2 + 3z + 1|$ vi) $g(z) = |-z^2 - z + 2|$

IX.C Constant and Linear Approximations

52. Find the constant approximation $A_0(x)$ and the linear approximation $A_1(x)$ of the following function $f(x)$ at the given point a. Then find the error functions $E_0(x)$ and $E_1(x)$:

    i) $f(x) = x^3 - 3x^2 + 2$; $a = 1$
    ii) $f(x) = x^3 - 3x^2 + 2$; $a = \frac{1}{2}$.

    In each case find numbers d_0 and d_1 so that

    $$|A_0(x) - f(x)| \leq \frac{1}{10} \quad \text{if} \quad |x - a| \leq d_0$$

    and

    $$|A_1(x) - f(x)| \leq \frac{1}{10} \quad \text{if} \quad |x - a| \leq d_1.$$

53. i) Find the constant approximation $A_0(x)$ and the linear approximation, $A_1(x)$ of the following functions $f(x)$ at the given points a. Then find the error functions $E_0(x)$ and $E_1(x)$:

    a) $f(x) = \frac{x^2-2}{x+1}$; $a = 0$ b) $f(x) = \frac{x^2-2}{x+1}$; $a = 10$

    c) $f(x) = x^3 - 6x^2 + 1$; $a = 1$ d) $f(x) = x^3 - 6x^2 + 1$; $a = .5$

    ii) In each case find a number d_0 so that

    $$|A_0(x) - f(x)| \leq \frac{1}{10} \quad \text{if} \quad |x - a| \leq d_0.$$

    iii) In each case find a number d_1 so that

    $$|A_1(x) - f(x)| \leq \frac{1}{10} \quad \text{if} \quad |x - a| \leq d_1.$$

54. A bakery produces cakes whose volume is given by $V(x) = x + \frac{2x}{x^2+1}$, where x is the amount of flour used. Thus 1 unit of flour produces cakes of 2 units of volume. Assuming that the volume of the cakes is to differ from 2 units by at most 1/10 unit, how accurately does the baker have to measure the flour?

55. Find the constant approximation $A_0(x)$ and the linear approximation $A_1(x)$ of the function

    $$f(x) = \frac{x^2 + 2x}{x - 1}$$

    at the point $a = 2$. Then find the error functions $E_0(x)$ and $E_1(x)$ and positive numbers d_0 and d_1 so that

    $$|A_0(x) - f(x)| \leq \frac{1}{2} \quad \text{if} \quad |x - a| \leq d_0$$

    and

    $$|A_1(x) - f(x)| \leq \frac{1}{20} \quad \text{if} \quad |x - a| \leq d_1.$$

56. Find the linear approximation of $\frac{1}{x-2}$ at the points $a = 1, 10, 100$. In each case find $d > 0$ so that $|E_1(x)| \leq 10^{-4}$ if $|a - x| \leq d$.

57. Find the constant approximation $A_0(x)$ and the linear approximation $A_1(x)$ of the following functions $f(x)$ at the given point a. Then find the error functions $E_0(x)$ and $E_1(x)$:

    i) $f(x) = x^3 + 2x^2 - 1 \quad ; a = -1$

    ii) $f(x) = \frac{1}{x^2-5} \quad ; a = 10$

    In each case find an upper bound for the absolute values $|E_0(x)|$ and $|E_1(x)|$ in the interval $|a - x| \le 1$.

58. Find the constant approximation $A_0(x)$ and the linear approximation $A_1(x)$ of the following functions $f(x)$ at the given point a. Then find the error functions $E_0(x)$ and $E_1(x)$:

    i) $f(x) = x^3 + 2x^2 - 1 \quad ; a = 1$

    ii) $f(x) = x^3 + 2x^2 - 1 \quad ; a = \frac{1}{2}$

    iii) $f(x) = \frac{x^2+2x}{x+1} \quad ; a = 2$

    iv) $f(x) = \frac{x^2+2x}{x+1} \quad ; a = -2.$

    In each case find numbers d_0 and d_1 so that

    $$|A_0(x) - f(x)| \le \frac{1}{10} \quad \text{if} \quad |x - a| \le d_0$$

    and

    $$|A_1(x) - f(x)| \le \frac{1}{10} \quad \text{if} \quad |x - a| \le d_1.$$

59. For each of the following, use IVT and/or Rolles' Theorem to determine the number of roots

    i) $f(x) = 2x^{12} + 2x^{10} + 3$

    ii) $f(x) = \frac{x^7}{7} + \frac{x^4}{2} + 2x + 1$

    iii) $f(x) = x^{10} - 2x - 1$

IX.D Newton's Method

60. Given that all the roots of P lie in the interval $[-R, R]$, what can you conclude about where the roots of P' lie?

61. For the given functions f, find an interval $[-R, R]$ containing all the roots of f.

    i) $f(x) = x^5 - x^3 + 4x^2 - 4$ ii) $f(x) = x^4 - 3x^2 - 4x - 3$

    iii) $f(x) = x^6 - 2x^4 - 4x^3 - 6x^2 - 4x - 3$ iv) $f(x) = 1 - \frac{x^2}{2} + \frac{x^4}{24}$

    v) $f(x) = x - \frac{x^3}{6} + \frac{x^5}{120}$

62. For the functions i), ii), iv), v) above divide the interval $[-R, R]$ into subintervals in which $f(x)$ either has no root or exactly one root.

63. Use Newton's method to find all the roots of the following functions to an accuracy of 10^{-6}.

    i) $x^3 - 3x - 3$ ii) $2x^3 - 3x^2 - 12x - 8.$

64. For each of the following functions $f(x)$ show that $f(x)$ has *exactly one* root in the given interval. Then, using Newton's method, find it to an accuracy of 10^{-6}.

    i) $f(x) = x^3 - 7x^2 + 7$; $[1, 2]$
    ii) $f(x) = 3x^3 + 2x + 1$; $[-1, 0]$.

65. For each function $f(x)$ below find how many zeros the function has in the interval $[-2, 2]$. When there is only one, find it using Newton's method to an accuracy of 10^{-5}:

    i) $f(x) = x^3 + x^2 + 1$ ii) $f(x) = x^3 + x + 1$
    iii) $f(x) = x^3 + 2x^2 + x + 1$

66. Use Newton's method to find all the roots of the following polynomials, to an accuracy of 10^{-5}:

    i) $x^5 - 4x^2 + 1$ ii) $x^5 + 4x^4 - 1$
    iii) $x^3 + 2x^2 + x + 1$ iv) $2x^3 - 3x^2 - 12x - 8$
    v) $x^2 - 3x + 1$ vi) $x^2 - 10x + 1$

67. For each root r of each polynomial in Problems 65, and for your choice of interval $[u, v]$ containing r find an i so that $|x_i - r| < 10^{-1,000,000,000,000,000}$. Do *not* attempt to get x_i.

    i) Let $f(x) = x^3 + \frac{1}{x}$. In each of the intervals $[1, 2]$, $[-2, -1]$ find a z (accurate within 10^{-4}) such that the EMVT holds.
    Hint: you may need Newton's method.
    ii) Find the linear approximation of f at the point $a = 4$, and find a constant K such that the error is $\le K$ in the interval $[2, 6]$.

68. Use Newton's method to find all the roots of the following functions to an accuracy of 10^{-6}.

    i) $f(x) = 2x^3 - 3x^2 - 12x - 8$ ii) $f(x) = x^3 - 3x^2 - 1$

69. For each of the following functions, divide the given interval into subintervals each of which contains at most one distinct root. When the subinterval contains a root, find a smaller subinterval still containing the root, and such that $f'(x)$ is never zero there. How many distinct roots does each $f(x)$ have on the given interval

    i) $f(x) = x^6 - 6x^4 + 9x^2 + 6$; $[-3, 4]$
    ii) $f(x) = 3x^5 - 5x^3 + 1$; $[-2, 2]$

70. Prove that there is exactly one solution of the equation

$$\frac{-x^2 + 2x - 3}{(x^2 - 3)^2} = -\frac{2}{3}$$

    in $[2, 3]$ and find it, using Newton's method to an accuracy of 10^{-6}.

71. If $f(x)$ is a rational function of order r show that $f(x)/f(-x) \to (-1)^r$ as $x \to +\infty$. Deduce that if $P(x)$ is a polynomial of odd degree then $P(x) = 0$ has at least one root.

IX.E Rational Functions as $x \to \pm\infty$

72. For each of the following rational functions $f(x)$

    i) State the order.

    ii) Evaluate $\lim_{x\to\infty} f(x)$.

    iii) Find a constant R and constants M and m such that (where $r = \text{order } f(x)$)

$$m|x|^r \le |f(x)| \le M|x|^r, \quad |x| \ge R$$

    a) $f(x) = \frac{x^2+1}{x}$ b) $f(x) = \frac{x}{x^2-1}$

    c) $f(x) = \frac{x^{50}}{x^{50}-1}$ d) $f(x) = \frac{12x^5+1}{3x^5-1}$

    e) $f(x) = \frac{2x}{3x^2-1}$ f) $f(x) = 1 + \frac{1}{3x^{20}-1}$

73. If f and g are rational functions and $\text{order}\, f > \text{order}\, g$, show that $\text{order}(f-g) = \text{order}\, f$.

74. i) Show that if $f(x)$ has odd order then there is a constant $C > 0$ such that

$$f(x) \quad \text{and} \quad f(-x) \quad \text{have opposite signs if } |x| \ge C.$$

    ii) Show that if $f(x)$ has even order then there is a constant C such that

$$f(x) \quad \text{and} \quad f(-x) \quad \text{have the same sign if } |x| \ge C.$$

    iii) Show that every polynomial of odd degree has at least one root.

75. If $f'(t) = Kf(t)$ where K is a nonzero constant, prove f is not a rational function.

76. Let $f(x) = \cos x + \sin x$, prove f is not a rational function.

77. Show that $\frac{\sqrt{1+x^2}}{x} \to 1$ as $x \to +\infty$ and $\frac{\sqrt{1+x^2}}{x} \to -1$ as $x \to -\infty$. Deduce that $\frac{\sqrt{1+x^2}}{x}$ is not a rational function. Is $\sqrt{1+x^2}$ a rational function?

IX.F Complex Numbers and Series

78. Let $z_1 = 7+9i$, $z_2 = -4+2i$, $z_3 = 3i$, $z_4 = 1+(\sqrt{3})i$, $z_5 = 2\pi - i$. For $\ell, m = 1,2,3,4,5$; $\ell \ne m$; find $z_\ell + z_m$, $z_\ell z_m$, $\bar{z}_\ell$, $z_\ell \bar{z}_m$, $|z_\ell|$, $\frac{1}{z_\ell}$.

79. Write each z_ℓ (from #78) in polar form.

80. For each of the following $z(t)$ find $z'(t)$

    i) $z(t) = (t^2+2t) + (4t^3-1)i$ ii) $z(t) = e^{2t+t^3 i}$

    iii) $z(t) = (\cos t)i$ iv) $z(t) = \frac{1}{t^3-(\tan t)i}$

81. Find $\int_0^t (e^u + u^2 i)\,du$.

82. Show that

    i) $\overline{re^{i\theta}} = re^{-i\theta}$ ii) $e^{\pi i} + 1 = 0$.

83. Show that every non-zero complex number has exactly two square roots.
(*Hint:* use polar form.)

84. Find all complex numbers z satisfying $z^3 = 1$ and $z^4 = 1$.

85. Find the limit of the sequence, if it exists

    i) $\left\{\left(\left(\frac{8}{7}\right)^n\right) i\right\}$

    ii) $\left\{\left(\frac{6n-5}{5n+1}\right) + \left(\frac{7-4n^2}{3+2n^2}\right) i\right\}$

    iii) $\left\{\left((-1)^{n+1}\frac{\sqrt{n}}{n+1}\right) + \left(6 - \left(\frac{5}{6}\right)^n\right) i\right\}$

    iv) $\left\{\left(\frac{n^2}{2^n}\right) + \left(n^{\frac{1}{n}}\right) i\right\}$

    v) $\left\{\left((-1)^n \frac{n^2}{2+n^2}\right) + \left(\left(\frac{1}{5}\right)^n\right) i\right\}$

86. Find the sum of the following series, if they converge.

    i) $\sum_{n=0}^{\infty} \left(\left(-\frac{1}{2}\right)+\left(\frac{2}{3}\right)i\right)^n$

    ii) $\sum_{n=0}^{\infty} \left(\left(\frac{1}{5}\right)^n + \left(\frac{3}{2}\right)^n i\right)$

87. Determine if the following series converge absolutely or diverge.

    i) $\sum_{n=1}^{\infty} \left(\left(\frac{1}{3+5^n}\right) + \left(\frac{7^n}{n!}\right)i\right)$

    ii) $\sum_{n=2}^{\infty} \left(\left(1+\frac{1}{n}\right)^n + \left(\frac{1}{n^n}\right)i\right)$

    iii) $\sum_{n=1}^{\infty} \left(\left(\frac{3^n+5}{5^n}\right) + \left(\frac{n^8}{2^n}\right)i\right)$

    iv) $\sum_{n=1}^{\infty} \left(\frac{(-1)^n}{1+3n^2}\right) i$

    v) $\sum_{n=2}^{\infty} \left(\frac{(-1)^n}{n \ln n}\right) i$

    vi) $\sum_{n=2}^{\infty} \left(\left(\frac{1}{n(\ln n)^2}\right) + \left(\frac{n}{2^n}\right)i\right)$

    vii) $\sum_{n=1}^{\infty} \left(n\left(\frac{3}{4}\right)^n\right) i$

    viii) $\sum_{n=1}^{\infty} (-1)^{n+1} \left(\left(\frac{\cos n}{n^2}\right) + \left(\frac{\sin n}{n^2}\right)i\right)$

    ix) $\sum_{n=1}^{\infty} \left(\frac{1}{1+\ln n}\right) i$

    x) $\sum_{n=1}^{\infty} \left(\left(\frac{n^2}{n^4+2n+3}\right) + i\right)$.