Contents

Chapter 0

REVIEW NOTES

0.0 Quadratic Equations and Completing the Square

Factor the following quadratic expressions

1. $x^2 - 3x, \quad 2x^2 + x, \quad \pi x^2 + 3\pi, \quad 17x^2 + 51x$ and $\sqrt{2}x^2 + 2x$
2. $x^2 - 9$ and $x^2 - 49$
3. $x^2 - 3$ and $x^2 - 1$
4. $x^2 - 2$ and $x^2 - \pi$
5. $x^2 + x + 1/4$ and $x^2 + x + 6$
6. $x^2 - x + 1/4$ and $x^2 - x - 6$
7. $x^2 + 2a + a^2$ and $x^2 + (a + b)x + ab$
8. $x^2 - 2x + 3$ and $x^2 - 3x - 40$
9. $3x^2 - 5x - 2$ and $8x^2 + 2x - 1$
10. $2x^2 - 5bx - 3b^2$ and $x^2 - 4x - 21$
11. $x^4 - 2x^2 + 1$ and $x^6 - 4x^3 - 21$

Solve the following equations by completing the square

12. $3x^2 + 6x - 1 = 0$
13. $3x(3x - 2) = 6x - 5$
14. $y^2 - 15y - 4 = 0$
15. $6u^2 + 7u - 3 = 0$
16. $x^2 - 2x + 9 = 0$
17. $4z^2 - 4z - 1 = 0$
18. $p(2p - 4) = 5$
19. $(x - 2)^2 + 3x - 5 = 0$
20. $(3x - 2)^2 + (x + 1)^2 - 0$
21. $5y^2 - 15y + 9 = 0$

Use the quadratic formula to solve the following equations

22. $5x^2 + 6x - 1 = 0$
23. $2x^2 = 18x + 5$
24. $x(2x - 3) = 2x - 6$
25. $6x^2 - 7x + 2 = 0$
26. $2x^2 = 13(x - 1) + 3$
27. $2x^2 - 6x - 1 = 0$
28. $1200y^2 = 10y + 1$
29. $x^2 + 2bx - c^2 = 0$
30. $x^2 - 6ax + 3a^2 = 0$
31. $\pi u^2 + (\pi^2 - 1)u - \pi = 0$
32. $x(x - \sqrt{2} + 4) = 4(x + 1)$
33. $3x^2 = 5(x - 1)^2$

For the following functions f find the discriminant of $f(x) = 0$ and determine whether the roots are real and unequal, real and equal, or do not exist. Graph $f(x)$ without plotting more than four points.

34. $f(x) = 4x^2 - 4x + 1$

35. $f(z) = z^2 + z + 1$

36. $f(x) = 4x^2 - x - 5$

37. $f(x) = 7x - 5x - 2$

38. $f(x) = x^2 + \sqrt{2}x + 1/4$

39. $f(x) = x^2 - ax - 1$

40. $f(x) = 3x^2 + \pi x + 4$

41. $f(x) = x^2 - 2ax + a^2$

42. $f(x) = \sqrt{3}x^2 - 2x - \sqrt{3}$

43. $f(x) = 9x^2 - 12x + 4$

Determine the values of K for which the equation has real and equal roots.

44. $5x^2 - 4x - (5 + K) = 0$

45. $(K + 2)x^2 + 3x + (K + 3) = 0$

46. $x^2 + 3 - K(2x - 2) = 0$

47. $(K + 2)x^2 + 5Kx - 2 = 0$

48. $x^2 - x(2 + 3K) + 7 = 0$

49. $(K - 1)x^2 + 2x + (K + 1) = 0$

Complete the square for each of the following functions

Examples:

a) $p(x) = x^2 + 2x + 10$

$p(x) = x^2 + 2x + 1 - 1 + 10$

$p(x) = (x + 1)^2 + 9$

b) $p(x) = x^2 - x$

$p(x) = x^2 - 2\,(1/2)\,x + (1/2)^2 - (1/2)^2$

$p(x) = x^2 - x + 1/4 - 1/4$

$p(x) = (x - 1/2)^2 - 1/4.$

50. $p(x) = x^2 + 5x + 2$

51. $p(x) = x^2 + 3x + 1$

52. $p(t) = -3t^2 - 5t + 1$

53. $p(t) = t^2 - 2t$

54. $p(x) = x^2 + 3x$

55. $p(x) = x^2 + 4x - 3$

56. $p(x) = 4x^2 + 12x + 10$

57. $p(x) = -16x^2 + 6x$

58. $p(x) = x^2 + 4b + c$

59. $p(x) = ax^2 + ax + b$

60. $p(x) = \pi(x^2 - 2x)$

61. $p(x) = 24(-x - 3x + 1)$

In the following questions write the given expression in the form $\sqrt{a}\,\sqrt{c^2 \pm (dx + b)^2}$ and simplify where c is a fraction.

Example: $\sqrt{-4x^2 + x}$

Let $q(x) = -4x^2 + x$

$$\begin{aligned}
q(x) &= -4\left(x^2 - \tfrac{x}{4}\right) \quad \text{complete square next:} \\
q(x) &= -4\left(x^2 - 2\left(\tfrac{1}{8}\right)x + \left(\tfrac{1}{8}\right)^2 - \left(\tfrac{1}{8}\right)^2\right) \\
q(x) &= -4\left(\left(x - \tfrac{1}{8}\right)^2 - \tfrac{1}{64}\right). \quad \text{Since } -4 = (-1)4 \text{ we get} \\
q(x) &= 4\left(\tfrac{1}{64} - \left(x - \tfrac{1}{8}\right)^2\right) \\
\sqrt{q(x)} &= 2\sqrt{\tfrac{1}{64} - \left(x - \tfrac{1}{8}\right)^2} = 2\sqrt{\tfrac{1-64\left(x-\frac{1}{8}\right)^2}{64}} \\
\sqrt{q(x)} &= {}^1\!/_4\sqrt{1 - 64\left(x - \tfrac{1}{8}\right)^2} = {}^1\!/_4\sqrt{1 - \left(8\left(x - \tfrac{1}{8}\right)\right)^2} \\
\sqrt{q(x)} &= {}^1\!/_4\sqrt{1 - (8x-1)^2}
\end{aligned}$$

62. $\sqrt{x^2 + 3x + 1}$
63. $\sqrt{-3t^2 + 5t + 1}$
64. $\sqrt{x^2 + 3x}$
65. $\sqrt{-16x^2 + 6x}$
66. $\sqrt{-x^2 + x + 1}$
67. $\sqrt{-x^2 + 2x - 4}$
68. $\sqrt{6x - 4x^2}$
69. $\sqrt{4x - x^2}$
70. $\sqrt{6 + x - 2x^2}$
71. $\sqrt{-x^2 - 2x + 8}$
72. $\sqrt{9x - 4x^2}$

0.1 Polynomials

Theorem I (Rational Root theorem): If an equation $f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_0$, $a_n \neq 0$ with *integral* coefficients, i.e. $a_i \in \mathbb{Z}$, has a nonzero rational root p/q in its lowest terms then p is a divisor of a_0 and q is a divisor of a_n.

Theorem II (Remainder theorem): Let $P(x)$ denote a polynomial of degree $n \geq 1$ and c a real number. Then there exists a polynomial $S(x)$ of degree $n - 1$ such that

$$P(x) = S(x)(x - c) + P(c).$$

Theorem III (Factor theorem): Let $P(x)$ denote a polynomial of degree $n \geq 1$ and c a real number. Then $x - c$ is a factor of $P(x)$ if and only if $P(c) = 0$, that is, if and only if c is a root of $P(x)$.

Example 1: Factor $f(x) = x^3 - x^2 - x - 2$

Solution:

By Theorem I the only possible rational roots are ± 1 and ± 2.

$f(1) = 1 - 1 - 1 - 2 = -3 \neq 0$
$f(-1) = 1 - 1 + 1 - 2 = -3 \neq 0$
$f(2) = 8 - 4 - 2 - 2 = 0$

Therefore 2 is a root of $f(x)$ and $x - 2$ is a factor of $f(x)$ by Theorem III.

$$\begin{array}{r|l}
 & x^2 + x + 1 \\
\hline
x - 2 & x^3 - x^2 - x - 2 \\
 & \underline{x^3 - 2x^2} \\
 & + x^2 - x \\
 & \underline{x^2 - 2x} \\
 & x - 2 \\
 & \underline{x - 2}
\end{array}$$

x^2+x+1 has no real factors since the discriminant $b^2-4ac=-3<0$. Therefore the answer is:

$$f(x)=(x-2)(x^2+x+1).$$

<u>Example 2</u>: Factor $f(x)=x^3+8$

The only possible rational roots are $\pm1,\ \pm2,\ \pm4$, or ±8. **(Theorem I)**

$f(-2)=0$ implies $x+2$ is a factor

$$\begin{array}{r|l} & x^2-2x+4 \\ \hline x+2 & x^3 \qquad\qquad +8 \\ & \underline{x^3+2x^2} \\ & \quad -2x^2 \\ & \quad \underline{-2x^2-4x} \\ & \qquad\qquad 4x+8 \\ & \qquad\qquad \underline{4x+8} \end{array}$$

Consider x^2-2x+4. The discriminant, b^2-4ac, is equal to -12. Thus x^2-2x+4 has no real factors. Therefore the answer is

$$f(x)=(x+2)(x^2-2x+4).$$

<u>Example 3</u>: Factor $P(x)=x^3+\frac{x^2}{6}-\frac{2x}{3}+\frac{1}{6}$.

Solution:

Since the coefficients of P are not all integers Theorem I cannot be applied directly. However $6P(x)=6x^3+x^2-4x+1$ and if we let $Q(x)=6x^3+x^2-4x+1$ then $6P(x)=Q(x)$ so that the roots of P and Q are the same. So we apply Theorem I to Q as follows: If p/q is a root of Q then p could be ±1 and q could be $\pm1,\ \pm2,\ \pm3$, or ±6. So the possibilities for p/q are ±1, $\pm\frac{1}{2}$, $\pm\frac{1}{3}$, or $\pm\frac{1}{6}$.

$$\begin{array}{rcll} Q(1) &=& 6+1-4+1 & \neq 0 \\ Q(-1) &=& -6+1+4+1 & =0 \end{array}$$

Therefore $x+1$ is a factor by Theorem III.

$$\begin{array}{r|l} & 6x^2-5x+1 \\ \hline X+1 & 6x^3+\ x^2-4x+1 \\ & \underline{6x^3+6x^2} \\ & \quad -5x^2-4x \\ & \quad \underline{-5x^2-5x} \\ & \qquad\qquad x+1 \end{array}$$

Now factor $6x^2-5x+1$. This quadratic $=6\left(x^2-\frac{5}{6}x+\frac{1}{6}\right)$. To factor the inner quadratic into $(x-\alpha)(x-\beta)$, find α, β from the formulas for the roots of a quadratic equation. Therefore $\alpha={}^1\!/\!_2\left({}^5\!/\!_6+\sqrt{({}^{25}\!/\!_{36})-({}^4\!/\!_6)}\right)$, $\beta={}^1\!/\!_2\left({}^5\!/\!_6-\sqrt{({}^{25}\!/\!_{26})-({}^4\!/\!_6)}\right)$. Therefore $6x^2-5x+1=6(x-{}^1\!/\!_3)(x-{}^1\!/\!_2)$

$$\begin{array}{rcl} Q(x) &=& 6(x+1)\left(x-\frac{1}{3}\right)\left(x-\frac{1}{2}\right) \\ P(x) &=& \frac{1}{6}\left(Q(x)\right)=(x+1)\left(x-\frac{1}{3}\right)\left(x-\frac{1}{2}\right) \end{array}$$

<u>Example 4</u>: Factor $f(x)=3x^4-7x^2+2$

Solution:

If p/q is a root of $f(x)$ then p could be ±1 or ±2 and q could be ±1 or ±3. So the only possibilities for p/q are ±1, $\pm{}^1\!/\!_3$, ±2, or $\pm{}^2\!/\!_3$.

$$\begin{aligned}
f(1) &= 3-7+2\neq 0\\
f(-1) &= 3-7+2\neq 0\\
f\left(\tfrac{1}{3}\right) &= \tfrac{1}{3^3}-\tfrac{7}{3^2}+2=\tfrac{1-21+54}{3^3}\neq 0\\
f\left(-\tfrac{1}{3}\right) &= f\left(\tfrac{1}{3}\right)\neq 0\\
f(2) &= 48-28+2\neq 0\\
f(-2) &= f(2)\neq 0\\
f\left(\tfrac{2}{3}\right) &= \tfrac{16}{3^3}-\tfrac{28}{3^2}+2=\tfrac{16-84+54}{3^3}\neq 0\\
f\left(-\tfrac{2}{3}\right) &= f\left(\tfrac{2}{3}\right)\neq 0
\end{aligned}$$

Therefore $f(x)$ has no *rational* roots but

$$\begin{aligned}
f(x) &= (3x^2-1)(x^2-2)\\
&= (\sqrt{3}x-1)(\sqrt{3}x+1)(x-\sqrt{2})(x+\sqrt{2}).
\end{aligned}$$

So the roots of $f(x)$ are irrational numbers $\frac{1}{\sqrt{3}}$, $-\frac{1}{\sqrt{3}}$, $\sqrt{2}$ and $-\sqrt{2}$.

In problems 73 to 85 factor the given polynomials as completely as possible.

73. $P(x)=2x^3+3x^2+4x-3$
74. $P(y)=2y^3+3y^2-8y+3$
75. $P(x)=2x^3+3x^2-6x+2$
76. $P(x)=x^4-5x^2-10x-6$
77. $P(x)=x^3-7x^2+8x+12$
78. $P(x)=x^3-27$
79. $P(x)=x^4-1$
80. $P(x)=\frac{x^3}{3}-\frac{x^2}{2}-\frac{x}{2}+\frac{1}{3}$
81. $P(x)=x^4-2x^3-4x^2-8x$
82. $P(x)=x^4+3x^3-x^2-3x$
83. $P(x)=-2x^4+7x^2-3$
84. $P(x)=x^2-4$
85. $P(x)=x^2-3$

0.2 Rational Functions

In the following problems express the rational functions in the form $f(x)/q(x)$ where f and q are polynomials. Then factor and simplify where possible.

86. $\frac{1}{y-x}\left(\frac{1}{x}-\frac{1}{y}\right)$: *Solution:* $\frac{1}{y-x}\left(\frac{y-x}{xy}\right)=\frac{1}{xy}$
87. $\frac{\frac{1}{x}-\frac{1}{y}}{x-y}$
88. $24xy\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{3}+\frac{1}{4}\right)$
89. $\left(\frac{2}{x}+\frac{1}{2}\right)+\left(\frac{1}{y}+\frac{1}{3}\right)$
90. $\frac{1}{x}-\frac{1}{2}-\frac{1}{y}-\frac{1}{3}$
91. $\frac{\frac{1}{x^2}-\frac{1}{3x}}{x-3}$
92. $\frac{1}{1+t^2}-\frac{2+(1+t)}{(1+t^2)^2}$
93. $\frac{x+\frac{1}{x-2}}{x-1}$
94. $\frac{x^2+\frac{4x^2+1}{x^2-2}}{(x^2+1)^2}$
95. $\frac{1}{36}\left(3x^2-\frac{3}{x^2}\right)^2+1$
96. $\frac{1}{4}\left(x^3-\frac{1}{x^3}\right)^2+1$
97. $1+\left(x^2-\frac{1}{4x^2}\right)^2$
98. $\left(x^2+\frac{1}{x}\right)\frac{1}{x-1}$
99. $\frac{1}{x^2}+\frac{1}{xy}$
100. $x-1+\frac{1}{2x}$

In the following problems perform the indicated division.

Example: $\frac{x^2+3x+1}{2x-1} = \frac{x}{2} + \frac{7}{4} + \frac{11}{4(2x-1)}$.

101. $\frac{(x-2)^2}{x}$

102. $\frac{4x^2+4x+1}{x}$

103. $\frac{5+t}{5-t}$

104. $\frac{x^2}{1-x^2}$

105. $\frac{3x-2}{2x+3}$

106. $\frac{4x+1}{3x-1}$

107. $\frac{x^2+1}{x^2-1}$

108. $\frac{x^2}{1-x^2}$

109. $\frac{-x^3}{x+3}$

110. $\frac{x^3-3}{x(x^2-9)}$

111. $\frac{x^3-3x+2}{x+3}$

112. $\frac{x^5+1}{x+1}$

113. $\frac{x^2+1}{x+1}$

114. $\frac{x^3-1}{x-1}$

In the following problems:

(a) Find all values of x where the function is not defined.

(b) Express the function $F(x)$ in the form $\frac{G(x)}{H(x)}$ where $G(x)$ and $H(x)$ are polynomials. Then factor and simplify where possible.

(c) Find for what values of x, if any, $F(x) = 0$.

(d) Find for what values of x $F(x) > 0$ and for what values of x $F(x) < 0$.

Example: $F(x) = -2x - 1 + \frac{x^2+x}{x+1}$.

Solution:

$$\begin{aligned} F(x) &= \frac{(x+1)(-2x-1)+x^2+x}{x+1} \\ F(x) &= \frac{-2x^2-x-2x-1+x^2+x}{x+1} \\ F(x) &= \frac{-x^2-2x-1}{x+1} \\ F(x) &= -\frac{(x+1)^2}{x+1} \end{aligned}$$

If $x \neq -1$ then $F(x) = -(x+1)$.

Note: $G(x) = -(x+1) \Rightarrow G(-1) = 0$ but $F(x) = -(x+1)^2/(x+1)$ is not defined at $x = -1$; i.e., you cannot divide by zero. Now we can answer the questions asked.

(a) $F(x)$ is not defined at $x = -1$.

(b) $F(x) = -\frac{(x+1)^2}{x+1}$ or $F(x) = -(x+1)$, $x \neq -1$.

(c) $F(x) = 0 \Rightarrow -\frac{(x+1)^2}{x+1} = 0$.

If $x - 1$ then $-(x+1)^2 = 0 \Rightarrow x+1 = 0 \Rightarrow x = -1$.

Contradiction! Therefore $F(x) \neq 0$ for any x.

(d) $F(x) > 0 \Rightarrow -\frac{(x+1)^2}{x+1} > 0 \Rightarrow \frac{(x+1)^2}{x+1} < 0$ but $(x+1)^2 > 0$ for all x. Hence $F(x) > 0$ if $x+1 < 0$; i.e. $\underline{F(x) > 0 \text{ if } x < -1}$.

$F(x) < 0$ for all x such that $F(x) \not> 0$ and $F(x) \neq 0$ that is $x > -1$.

115. $x - 4 + \frac{4}{x}$
116. $4x + 4 + \frac{1}{x}$
117. $\frac{10}{5-t} - 1$
118. $\frac{1}{1-x^2} - 1$
119. $\frac{3}{2} - \frac{3}{2x+3}$
120. $\frac{4}{3} + \frac{7}{3(3x-1)}$
121. $1 + \frac{2}{x^2-1}$
122. $\frac{1}{1-x^2} - 1$
123. $\frac{27}{x+3} - x^2 + 3x - 9$
124. $\frac{9x-3}{x^3-9x} + 1$
125. $x^2 - 3x + 6 - \frac{16}{x+3}$

0.3 Absolute Values and Inequalities

Definition: The absolute value of a number z, written $|z|$, is defined by

$$|z| = \begin{cases} z, & \text{if } z \geq 0 \\ -z, & \text{if } z < 0 \end{cases}$$

Examples: $|5| = 5$; $|-3| = -(-3) = 3$; $|0| = 0$; $|(-1)^n| = 1$ for every $n \in \mathbb{Z}$.

$$|x-1| = \begin{cases} x-1 & \text{if } x-1 \geq 0 \\ -x+1 & \text{if } x-1 < 0 \end{cases} = \begin{cases} x-1 & \text{if } x \geq 1 \\ -x+1 & \text{if } x < 1 \end{cases}$$

$$|\sin\theta| = \begin{cases} \sin\theta & \text{if } \sin\theta \geq 0 \\ -\sin\theta & \text{if } \sin\theta < 0 \end{cases}$$

Properties of absolute value

1. $|x| \geq 0$; $|x| = 0$ if and only if $x = 0$.
2. $|xy| = |x||y|$; $|x/y| = |x|/|y|$ $(y \neq 0)$.
3. $|-x| = |x|$; $|x-y| = |y-x|$.
4. $|x|^2 = x^2$.
5. $\sqrt{x^2} = |x|$. (This is **very** important. Many commercial software packages have this wrong!)
6. If $a > 0$ then $|x| < a$ is equivalent to $-a < x < a$ and also equivalent to $x \in (-a, a)$.
7. If $a \geq 0$ then $|x| \leq a$ is equivalent to $-a \leq x \leq a$ or $x \in [-a, a]$ and also equivalent to $x \in [-a, a]$.
8. If $a > 0$, b any number, then $|x - b| < a$ is equivalent to $b - a < x < b + a$ and also equivalent to $x \in (b - a, b + a)$.

Note: $\sqrt{x^2} = |x|$.

The symbol $\sqrt{z}$ is defined for $z \geq 0$ and means the *positive* square root of the number z.

Examples:

1. $\sqrt{5^2} = \sqrt{25} = |5| = 5$

   $\sqrt{(-5)^2} = \sqrt{25} = |-5| = 5$

2. $\sqrt{x^2 + 2x + 1} = \sqrt{(x+1)^2} = |x+1|$

   $= \begin{cases} x+1, & \text{if } x+1 \geq 0 \\ -x-1, & \text{if } x+1 < 0 \end{cases}$

   $= \begin{cases} x+1, & \text{if } x \geq -1 \\ -(x+1), & \text{if } x < -1 \end{cases}$

3. Compute the distance between two points on the number line.

   Solution: Let the points be $(a, 0)$ and $(b, 0)$ in the coordinate plane. Let the distance between the points be s. Then

   $$s = \sqrt{(a-b)^2 + (0-0)^2} = |a-b|.$$

   If $a - b > 0$; i.e., $a > b$ then $s = a - b$. If $a - b < 0$; i.e. $a < b$ then $s = -(a-b)$ or $s = b - a$.

4. $\sqrt{x^4} = \sqrt{(x^2)^2} = |x^2|$

   $|x^2| = \begin{cases} x^2 & \text{if } x^2 > 0 \\ -x^2 & \text{if } x^2 < 0 \end{cases}$

   But by example 3 x^2 cannot be less than zero for any value of x. Hence $|x^2| = x^2$.
   $\therefore \sqrt{x^4} = x^2$ for all values of x.

5. $x^2 > 3 \Rightarrow \sqrt{x^2} > \sqrt{3}$ because the square root function is a strictly increasing function so $|x| > \sqrt{3}$. There are two cases:

   I. If $x \geq 0$ then $|x| = x \Rightarrow x > \sqrt{3}$.

   For case I, $x^2 > 3$ solution set is $(0, \infty) \cap (\sqrt{3}, \infty) = (\sqrt{3}, \infty)$.

   II. If $x < 0$ then $|x| = -x \Rightarrow -x > \sqrt{3} \Rightarrow x < -\sqrt{3}$.

   For case II, $x^2 > 3$ solution set is $(-\infty, -\sqrt{3}) \cap (-\infty, 0) = (-\infty, -\sqrt{3})$.

   Solution set for $x^2 > 3$ is the union of both cases; i.e., $x \in (-\infty, -\sqrt{3}) \cup (\sqrt{3}, \infty)$.

6. Solve for x where y is defined by $y = x^2$, $x \le 0$.

   Solution:

$$\begin{aligned} y &= x^2 \\ \therefore \sqrt{y} &= \sqrt{x^2} \\ |x| &= \sqrt{y} \\ |x| &= -x \quad \text{since } x \le 0 \\ \therefore -x &= \sqrt{y} \\ x &= -\sqrt{y} \end{aligned}$$

In the following problems solve the given inequalities.

Example 1: Solve $|x-3| > 2x+1$

Solution:

Case 1. If $x - 3 \ge 0$ or $x \ge 3$

then $|x-3| > 2x+1$

$x - 3 > 2x + 1$

$-2x + x > 1 + 3$

$-x > 4$

$x < -4 \qquad \text{If } x \ge 3.$

This is a contradiction. Therefore the solution set for case 1 is the empty set $\emptyset$.

Case 2. If $x - 3 < 0$ or $x < 3$

then $|x-3| > 2x+1$

$-(x-3) > 2x+1$

$-x - 2x > 1 - 3$

$-3x > -2$

$3x < 2$

$x < \frac{2}{3}$ if $x < 3$.

Therefore $|x-3| > 2x+1$ if

$x \in (-\infty, 2/3) \cap (-\infty, 3) = (-\infty, 2/3)$.

Example 2: Solve $|x-3| > |2x+1|$.

Solution: First note if a and b are nonnegative real numbers then $a^2 < b^2 \Leftrightarrow a < b$.
IT IS NOT TRUE IF a OR b IS NEGATIVE; i.e., $2^2 < (-3)^2$ but $2 > -3$.
$\therefore |x-3| > |2x+1|$ is equivalent to

$$|x-3|^2 > |2x+1|^2$$
$$x^2 - 6x + 9 > 4x^2 + 4x + 1$$
$$-3x^2 - 10x > -8$$
$$3x^2 + 10x < 8$$
$$3x^2 + 10x - 8 < 0$$
$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-10 \pm \sqrt{100 + 96}}{6} = \frac{-10 \pm 14}{6} = \frac{-5 \pm 7}{3}$$
$$x = \frac{2}{3} \text{ or } -4$$
$$\therefore \left(x - \frac{2}{3}\right)(x+4) < 0.$$

Case 1. $\left(x - \frac{2}{3}\right) < 0$ and $x + 4 > 0$
$x < \frac{2}{3}$ and $x > -4$
$x \in \left(-4, \frac{2}{3}\right)$.

Case 2. $\left(x - \frac{2}{3}\right) > 0$ and $x + 4 < 0$
$x > \frac{2}{3}$ and $x < -4$.
This case is not possible.
Therefore $|x - 3| > |2x + 1|$ for $x \in \left(-4, \frac{2}{3}\right)$.

Example 3: Solve the inequality $\frac{x}{x-1} \geq \frac{2}{x-1}$.

Solution: Case 1. if $x - 1 > 0 \Rightarrow x > 1$ then $x \geq 2$. Therefore the solution set is $x > 1$ and $x \geq 2 = x \geq 2$.

Case 2. if $x - 1 < 0 \Rightarrow x < 1$ then $x \leq 2$ because multiplying both sides of the inequality by a quantity less than zero reverses the inequality. So $x < 1$ and $x \leq 2 = x < 1$ is the solution set for this case.
The solution set for $\frac{x}{x-1} \geq \frac{2}{x-1}$ is the union of the two cases. ie. $x < 1 \cup x \geq 2$.

In the following problems solve the given inequalities.

126. $(x - 1)|x + 1| > 0$
127. $|x^2 - 8x + 16| > 0$
128. $|x^2 + 2x + 1| < 0$
129. $|(x - 2)^2| > 0$
130. $|(x - \pi)^2| < 0$
131. $x|x| > 0$ and $x|x| < 0$
132. $x|x - 1| > 0$ and $x|x - 1| < 0$
133. $x^2|x| > 0$
134. $(x^2 - 1)|x| > 0$
135. $(x^2 + 1)|x| > 0$
136. $\sqrt{3x^2} > 0$
137. $x^2\sqrt{x^2} > 0$
138. $x\sqrt{x^2} > 0$ and $x\sqrt{x^2} < 0$
139. $\sqrt{x^2 - 6x + 9} > 1$
140. $\sqrt{4x^2 + 12x + 9} > 2$
141. $\sqrt{(2x + 3)^2} > 0$
142. $\sqrt{|x|} > 0$
143. $\sqrt{|x| - 2} > 0$
144. $\sqrt{2|x| - 1} > 0$
145. $|x| - \pi > 0$; and $|x| - \pi < 0$
146. $3|x| + 1 > 0$; and $3|x| + 1 < 0$
147. $|x + 3| - 3 < 0$; and $|x + 3| - 3 > 0$
148. $|x^2 - 2| + 1 < 0$; and $|x^2 - 2| + 1 > 0$
149. $|x^2 + 1| + 1 < 0$; and $|x^2 + 1| + 1 > 0$
150. $\frac{3}{x} < 5$
151. $\frac{2}{x-1} < 4$
152. $\frac{4}{x} < \frac{5}{3}$
153. $\frac{4}{x} + \frac{3}{x} < 0$
154. $\frac{x}{3-x} < 2$
155. $\frac{x^2}{3-x} < 2$
156. $\frac{x}{(3-x)^2} < 2$
157. $\frac{x^2}{x} > 1$
158. $\frac{x-1}{x} < 4$
159. $\frac{2}{x} - 3 < \frac{4}{x} + 1$
160. $2x + \frac{6-3x}{4} < 4$
161. $\frac{x+3}{x-2} < 5$
162. $\frac{2x-3}{x+2} < \frac{1}{3}$
163. $\frac{x+3}{x-4} < -2$
164. $\frac{x-3}{x-1} > x - 4$
165. $\frac{x^2-2}{x+\sqrt{2}} > -4$
166. $\frac{x^2-3x+2}{x-2} > 0$
167. $\frac{x-2}{x+3} < \frac{x+1}{x}$

0.4 Quadratic Inequalities

In the following problems determine for what values of x the function is greater than zero and for what values of x the function is less than zero. Express your answers in interval notation.

Example: Find for what values of x, $f(x) > 0$, and for what values of x, $f(x) < 0$, where $f(x) = x^2 + x - 2$.

First factor $f(x)$; i.e., $f(x) = (x - 1)(x + 2)$.

(a) $f(x) > 0 \Leftrightarrow (x - 1)(x + 2) > 0$

A product $a \cdot b$ of real numbers is positive (> 0) if and only if both factors are nonzero and have the same sign. Therefore, the inequality $(x - 1)(x + 2) > 0$ is satisfied if and only if

$$[(x - 1) > 0 \quad \text{and} \quad (x + 2) > 0] \quad \text{or} \quad [(x - 1) < 0 \quad \text{and} \quad (x + 2) < 0].$$

Observe that any x_1 that is a solution of $(x - 1) > 0$ **and** $(x + 2) > 0$ must be a solution of $(x - 1) > 0$ as well as a solution of $(x + 2) > 0$. Therefore in this context 'and' has the same meaning as intersection in set theory. On the other hand $A > 0$ **or** $B < 0$ means x_1 is in this solution set if x_1 is a solution of $A > 0$ alone, and need not satisfy $B < 0$; or if x is a solution of $B < 0$ alone and does not necessarily satisfy $A > 0$; or if x satisfies both $A > 0$ and $B < 0$. In terms of set theory A **or** B is equivalent to $A \cup B$.

$x - 1 > 0 \Leftrightarrow x > 1 \Leftrightarrow x \in (1, \infty)$

$x + 2 > 0 \Leftrightarrow x > -2 \Leftrightarrow x \in (-2, \infty)$

Therefore: $(x > 1 \text{ and } x > -2) \Leftrightarrow x \in (1, \infty) \cap (-2, \infty) \Leftrightarrow x \in (1, \infty)$

$x - 1 < 0 \Leftrightarrow x < 1 \Leftrightarrow x \in (-\infty, 1)$

$x + 2 < 0 \Leftrightarrow x < -2 \Leftrightarrow x \in (-\infty, -2)$

$(x < 1 \text{ and } x < -2) \Leftrightarrow x \in (-\infty, 1) \cap (-\infty, -2) \Leftrightarrow x \in (-\infty, -2)$

Therefore the solution set for $f(x) > 0$ is $x \in (-\infty, -2) \cup (1, \infty)$.

(b) $f(x) < 0 \Leftrightarrow (x - 1)(x + 2) < 0$

and this is the same as saying:

$$[(x - 1) < 0 \text{ and } (x + 2) > 0] \text{ or } [(x - 1) > 0 \text{ and } (x + 2) < 0].$$

$(x - 1) < 0 \Leftrightarrow x < 1 \Leftrightarrow x \in (-\infty, 1)$

$(x + 2) > 0 \Leftrightarrow x > -2 \Leftrightarrow x \in (-2, \infty)$

Therefore: $(x < 1 \text{ and } x > -2) \Leftrightarrow x \in (-\infty, 1) \cap (-2, \infty) \Leftrightarrow x \in (-2, 1)$

$(x - 1) > 0 \Leftrightarrow x > 1 \Leftrightarrow x \in (1, \infty)$

$(x + 2) < 0 \Leftrightarrow x < -2 \Leftrightarrow x \in (-\infty, -2)$

Therefore: $(x > 1 \text{ and } x < -2) \Leftrightarrow x \in (1, \infty) \cap (-\infty, -2) = \emptyset$.

(Note $\emptyset$ is the empty set; i.e., no real number can be a member of both $(1, \infty)$ and $(-\infty, -2)$, hence the solution set is empty.)

Combining the above we find that $f(x) < 0$ iff $x \in (-2,1) \cup \emptyset \Leftrightarrow \underline{x \in (-2,1)}$. Thus the solution set is the open interval $(-2,1)$.

Let us look at these solution sets on the number line. Denote where $f(x) > 0$ by ++++++ and where $f(x) < 0$ by − − − − −

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++++++++) ( − − − − − − − ) (++++++++
_____________________________________
       −2          0      1
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The two missing points, -2 and 1, are those x for which $f(x) = 0$, as only for these two values of x is $f(x)$ neither greater nor smaller than zero.

168. $f(x) = x^2 - 2x - 3$
169. $f(x) = x^2 - 5x + 6$
170. $f(x) = x^2 + x - 42$
171. $f(x) = 2x^2 - x - 1$
172. $f(x) = 4x^2 + 4x + 1$
173. $f(x) = x^2 - 9$
174. $f(x) = x^2 - 3$
175. $f(x) = x^2 - 4x$
176. $f(x) = 16x^2 - 2x$
177. $f(x) = x^2 + 3x$
178. $f(x) = 2x^2 - x$
179. $f(x) = x^2 - a^2$
180. $f(x) = x^2 + x + 1$

Solution for problem #180:

$f(x) = x^2 + x + 1/4 - 1/4 + 1$ (complete square)

$f(x) = (x + 1/2)^2 + 3/4$

$\therefore f(x) > 0 \Leftrightarrow (x + 1/2)^2 + 3/4 > 0$

$\Leftrightarrow (x + 1/2)^2 > -3/4$

Since $(x + 1/2)^2 \geq 0$ for every $x \in R$, $(x + 1/2)^2 > -3/4$ is true for every $x \in R$. Therefore $f(x) > 0$ for $x \in (-\infty, \infty)$. As for $f(x) < 0$, there can be no solutions as we already saw that every real x satisfies $f(x) > 0$! Indeed, $f(x) < 0 \Leftrightarrow (x + 1/2)^2 < -3/4$. Since $(x + 1/2)^2 > 0$ for every $x \in R$, the solution set for $f(x) < 0$ is $\emptyset$, the empty set.

181. $f(x) = x^2 + 1$
182. $f(x) = 3x^2 + 2$
183. $f(x) = x^2$
184. $f(x) = -x^2$
185. $f(x) = -x^2 + 2x - 2$
186. $f(x) = -x^2 - 4$
187. $f(x) = x^2 + 3x + 3$
188. $f(x) = 2x^2 + x + 1$

0.5 The Chart Method for Inequalities

A. Factor the numerator and denominator of the given expression.

B. Simplify:

1) The sign of an expression is unaffected by any factor that is never negative. Therefore it may be omitted in the chart providing you record the value of x that makes the expression zero or undefined on the number line.
2) The sign of an expression is unaffected by cancelling a factor that appears in the numerator and denominator. Therefore omit on the chart but record on the number line the value of x that makes the factor zero. The expression is undefined for this value of x.

3) Recall the following facts.
   (a) $e^x > 0$ for all $x \in \mathbb{R}$.
   (b) $\sqrt{x} \geq 0$ and $\sqrt{x}$ is defined only on the interval $x \geq 0$. Therefore any expression containing this factor is restricted to a subset of the interval $x \geq 0$.
   (c) $\ln x$ is defined only on the interval $x > 0$. Hence any expression containing $\ln x$ as a factor is restricted to a subset of the interval $x > 0$.
   (d) $ax^2 + bx + c$. If $a > 0$ and $b^2 - 4ac < 0$ then $ax^2 + bx + c > 0$ for all $x \in \mathbb{R}$. If $a < 0$ and $b^2 - 4ac < 0$ then $ax^2 + bx + c < 0$ for all $x \in \mathbb{R}$.

C. Analyze each factor to determine where it is negative, zero, positive or undefined.

D. Construct a chart.

1) Draw a number line and circle each point on this line where the expression is undefined.
2) Draw a horizontal line for each factor appearing in the numerator or denominator above the number line. Label each line.
3) Mark each place on the line where the factor changes sign. Draw a vertical line through these points and record their values on the number line.
4) For each factor, the line is dashed where the factor is negative and solid where the factor is positive.
5) Step 3 partitions the number line into intervals. For each interval count the number of dashed lines. If the count is even the expression is positive on this interval. If the count is odd then the expression is negative in the interval. Record this information above the last line in the interval with a (+) or a (−).

E. Use the chart to answer the question asked. Write your answer in interval notation.

Example: Solve the inequality:

$$\frac{(x-1)(x^4+x^3)}{x^2-3x+2} < 0.$$

Solution: Let

$$F(x) = \frac{(x-1)(x^4+x^3)}{x^2-3x+2}.$$

a.) Factor

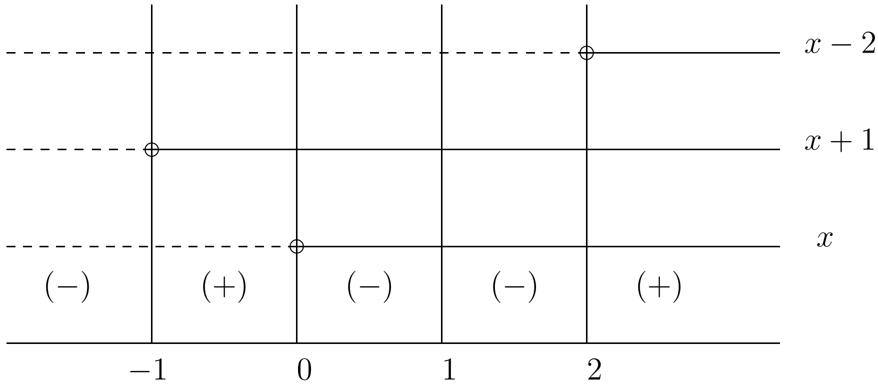
$$F(x) = \frac{x^2(x)(x+1)(x-1)}{(x-2)(x-1)}.$$

b.) $F(x)$ is undefined at $x = 1$ and $x = 2$. The sign of $F(x)$ is the same as the sign of $\frac{x(x+1)}{(x-2)}$.

c.)

FACTOR	(−)	(0)	(+)	UNDEFINED
x	$x < 0$	$x = 0$	$x > 0$	never
$x + 1$	$x < -1$	$x = -1$	$x > -1$	never
$1/(x-2)$	$x < 2$	never	$x > 2$	2

d.)



e.) $F(x) < 0$ for $x \in (-\infty, -1) \cup (0, 1) \cup (1, 2)$.

In the following problems solve the given inequality

189. $\frac{6+x-x^2}{(x^2+x+1)(x+4)(x-6)^2} \geq 0$

190. $\frac{(x^2+6x+9)(x^3+x^2)}{3x-5} < 0$

191. $x(x-3)(6-x) < 0$

192. $\frac{(x^2-5x+4)(x+2)}{(x^2+3)(2x+1)} \geq 0$

193. $\frac{(x+1)(2x-3)}{x+5} \geq 0$

194. $\frac{(2-5x)(x^2-3x-4)}{(x^2+2x+2)(x+2)} \leq 0$

195. $\frac{(x^4+4)(x^2-1)}{(2x+1)^2(x-4)} > 0$

196. $\frac{(x^2+2x+1)(3x+1)}{(2x^2-6x-20)(1-x)} \leq 0$

197. $\frac{x^4(4-x)(x^2-x-6)}{(x+3)^2(x^2-25)} \leq 0$

198. $\frac{(x^2+1)(x^4+1)}{(x^2+2)(x^6+6)} > 0$

199. $\frac{x^5(x^2+1)(x^2-4)^2}{-(x-1)^2} > 0$

200. $\frac{|x-1|(x^2-2)}{(x-1)} > 0$

201. $\frac{|x^3-2|(x^2-x-6)}{x^2-25} \leq 0$

202. $\frac{e^x(x-1)(x+2)}{x} < 0$

203. $\frac{(x^2+6x+9)\ln x}{x^3} > 0$

204. $\frac{(x^2-1)\ln|x|}{x+1} \leq 0$

205. $\ln(x+1) - \ln(x-1) + \ln(x^2-3x+2) > 0$

206. $\ln\left[\frac{(x^2+1)(x-3)}{x}\right] > 0$

207. $\frac{(x^2-5)^2}{(x^4-4)(x+3)} \geq 0$

0.6 Exponents

If a and b are real numbers, $a > 0$, $b > 0$ and α and β are rational numbers then

1. $a^0 = 1$
2. $(a^\alpha)^\beta = a^{\alpha\beta}$
3. $a^{\alpha+\beta} = a^\alpha a^\beta$
4. $a^{-\alpha} = \frac{1}{a^\alpha}$
5. $(ab)^\alpha = a^\alpha b^\alpha$
6. $(a^\alpha)^{\frac{1}{\beta}} = \left(a^{\frac{1}{\beta}}\right)^\alpha = a^{\frac{\alpha}{\beta}}$

In general the above laws do not hold when $a < 0$ and α, β are rational numbers.

Examples: $\left[(-2)^2\right]^{\frac{1}{2}} \neq \left[(-2)^{\frac{1}{2}}\right]^2 \neq (-2)^1$

$[(-3)(-4)]^{\frac{1}{2}} \neq (-3)^{\frac{1}{2}}(-4)^{\frac{1}{2}}$.

In the following problems write the given expression in the form p/q where p and q are integers.

Example: $4^{\frac{1}{2}} + 4^{-\frac{1}{2}} = 4^{\frac{1}{2}} + \frac{1}{4^{\frac{1}{2}}} = 2 + \frac{1}{2} = \frac{5}{2}$.

208. (a) $\frac{3^{-2}}{2^{-3}}$ (b) $\frac{1}{2^{-1}}$ (c) $\left(\frac{3}{5}\right)^{-1}$ (d) $\left(-\frac{1}{3}\right)^{-2}$

209. (a) $\frac{2^0}{3^{-2}}$ (b) $\frac{5^{-1}}{3^{-2}}$ (c) $(-8)^{-\frac{1}{3}}$ (d) $(16)^{-\frac{1}{4}}$

210. (a) $3^{-2} + 3$ (b) $5^{-1} + 25^0$ (c) $16^{-\frac{1}{2}} - 16^{\frac{1}{4}}$ (d) $8^{-\frac{1}{3}} - 2^0$

211. (a) $\frac{16^{\frac{1}{2}}}{8^{-\frac{2}{3}}}$ (b) $4^{-1} + 3^{-1}$ (c) $\left(\frac{1}{5}\right)^{-1} - \left(\frac{1}{7}\right)^{-1}$

Assume that all variables represent positive real numbers only. Write each of the following as a product or quotient of powers in which each variable occurs but once, and all exponents are positive.

Examples: $\left(\frac{x^{-1}y^2z^0}{x^3y^{-4}z^2}\right)^{-1} = \frac{xy^{-2}z^0}{x^{-3}y^4z^{-2}} = \frac{x^4z^2}{y^6}$.

212. (a) $x^{-3}x^5$ (b) $(x^2y^{-3})^{-1}$ (c) $\frac{x^5}{x^{-2}}$

213. (a) $(x^{-3})^2$ (b) $(x^{\frac{1}{2}})^{-3}$ (c) $(x^3)^{-\frac{1}{3}}$

214. (a) $(x^2y^{-1})^{-\frac{1}{2}}$ (b) $(x^3y^{-2})^{-\frac{1}{6}}$ (c) $(x^{-2}y^3)^0$

215. (a) $\frac{x^{-1}}{y^{-1}}$ (b) $\frac{x^{-2}}{y^{-3}}$ (c) $\frac{a^2x^{-3}}{b^2y^{-2}}$

216. (a) $\frac{a^{-2}b^{-2}c}{ab^{-3}c^0}$ (b) $\left(\frac{x^{-1}y^3}{2x^0y^{-5}}\right)^{-2}$ (c) $\left(\frac{a^{-1}b^{-2}}{3^0ab}\right)^{-1}$

In the following problems write the given expressions as a single fraction involving positive exponents only. Assume that all variables represent positive real numbers only.

217. (a) $x^{-1} + y^{-1}$ (b) $x^{-1} - y^{-1}$ (c) $\frac{x+(xy)^{-1}}{x}$

218. (a) $x^{-1} + y^{-2}$ (b) $(x^{-1} + x^{-2})^{-1}$ (c) $x^{-1} + \frac{1}{x^{-1}}$

219. (a) $a^{-2} + b^{-2}$ (b) $\frac{x^{-1}}{y^1} + \frac{y}{x}$ (c) $\frac{r}{s^{-1}} + \frac{r^{-1}}{s}$

220. (a) $(x+y)^{-1}$ (b) $(a-b)^{-2}$ (c) $xy^{-1} + x^{-1}y$

221. (a) $x^{-1}y - xy^{-1}$ (b) $\frac{x^{-1}+y^{-1}}{(xy)^{-1}}$ (c) $\frac{a}{b^{-1}} + \left(\frac{a}{b}\right)^{-1}$

222. (a) $(x^{-1} - y^{-1})^{-1}$ (b) $\frac{x^{-1}+y^{-1}}{x^{-1}-y^{-1}}$ (c) $\frac{x^{-1}-y^{-1}}{x^{-1}+y^{-1}}$

Compute the following expressions if they are defined. Do *not* write the answer as a decimal approximation.

Examples:

(1) $(-3)^{\frac{1}{2}}$ not defined.

(2) $\frac{\sqrt{300}}{\sqrt{8}} = \frac{\sqrt{100}\ \sqrt{3}}{\sqrt{4}\ \sqrt{2}} = \frac{5\sqrt{3}}{\sqrt{2}} = \frac{5\sqrt{6}}{\sqrt{2}}$.

223. (a) $\left(\frac{8}{27}\right)^{\frac{2}{3}}$ (b) $(32)^{\frac{1}{5}}$ (c) $(-27)^{\frac{1}{3}}$

224. (a) $(27)^{\frac{2}{3}}$ (b) $32^{\frac{3}{5}}$ (c) $(125)^{\frac{2}{3}}$

225. (a) $(-8)^{\frac{4}{3}}$ (b) $(-64)^{\frac{2}{3}}$ (c) $(-125)^{\frac{2}{4}}$

226. (a) $(-4)^{\frac{10}{2}}$ (b) $(-4)^{\frac{3}{2}}$ (c) $[(-8)(-2)]^{\frac{1}{2}}$

227. (a) $\frac{\sqrt{32}}{\sqrt{200}}$ (b) $\frac{\sqrt{27}}{-\sqrt{160}}$ (c) $\frac{\sqrt{20}}{\sqrt{6}}$

In the following problems find the factor A.

Example: $y^{-\frac{1}{2}} + y^{\frac{1}{2}} = Ay^{-\frac{1}{2}}$

$A = (1+y)$. Check: $y^{-\frac{1}{2}}(1+y) = y^{-\frac{1}{2}} + y^{\frac{1}{2}}$.

228. (a) $y^{\frac{3}{4}} = Ay^{\frac{1}{4}}$ (b) $x^{\frac{3}{5}} = Ax^{\frac{1}{5}}$

229. (a) $x^{-\frac{1}{3}} = Ax^{-\frac{2}{3}}$ (b) $y^{-\frac{1}{4}} = Ay$

230. (a) $x^{\frac{3}{2}} + x = Ax$ (b) $y^{\frac{1}{2}} + y = Ay$

231. (a) $x - x^{\frac{2}{3}} = Ax^{\frac{1}{3}}$ (b) $a^{\frac{2}{3}} + a^{\frac{1}{3}} = Aa$

232. (a) $x^{\frac{1}{2}} + x^{\frac{3}{2}} = Ax^{\frac{3}{2}}$ (b) $x^{-\frac{3}{2}} + x^{-\frac{1}{2}} = Ax^{-\frac{1}{2}}$

Write each of the following as a single fraction in which the denominator is rationalized and simplify where possible.

Example: $\frac{1}{\sqrt{x}-\sqrt{y}}$ $(x \neq y)$

$\frac{\sqrt{x}+\sqrt{y}}{(\sqrt{x}-\sqrt{y})(\sqrt{x}+\sqrt{y})} = \frac{\sqrt{x}+\sqrt{y}}{x-y}$ $(x \neq y)$

233. (a) $3/(\sqrt{2}-1)$ (b) $-4/(1+\sqrt{3})$ (c) $1/(2-\sqrt{2})$

234. (a) $(\sqrt{x}+\sqrt{y})/(\sqrt{x}-\sqrt{y})$ (b) $\sqrt{x}/(\sqrt{x}-\sqrt{y})$ (c) $\frac{\sqrt{x+a}}{1-\sqrt{x+a}}$

235. (a) $\sqrt{x+1} - \frac{x}{\sqrt{x+1}}$ (b) $\sqrt{x^2-2} - \frac{(x^2+1)}{\sqrt{x^2-2}}$

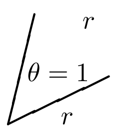
236. (a) $\frac{x}{\sqrt{x^2+1}} - \frac{\sqrt{x^2+1}}{x}$ (b) $\frac{x}{\sqrt{x^2-1}} + \frac{\sqrt{x^2-1}}{x}$

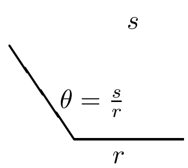
0.7 Trigonometry

0.7.1 Radian Measure

Definition.

An angle of one radian, when placed with its vertex at the center of a circle, intercepts an arc which is equal in length to the radius.





It follows from the above definition that the radian measure θ of an angle whose vertex is at the center of a circle is equal to the ratio of the length of the intercepted arc to the radius; thus

$$\theta = \frac{s}{r}$$

where s, the length of the intercepted arc, and r, the radius, are given in the same units of length.

NOTES:

(a) Radian measure will be used in your calculus and physics courses.

(b) θ is a number. If we were to use angles measured in degrees the unit degree would have to be used.

(c) The ratio of the circumference of a circle to its radius is 2π; hence $360°$ corresponds to 2π and $180°$ corresponds to π.

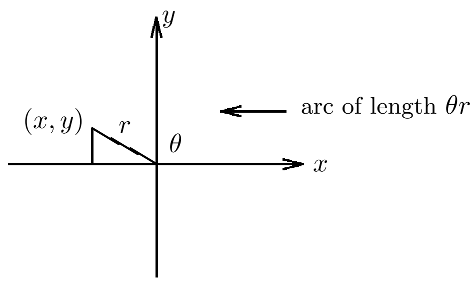
(d) If d is the number of degrees in a given angle and θ is the number of radians in the same angle, then

$$\frac{d}{180} = \frac{\theta}{\pi} \qquad \text{or} \qquad \theta = \frac{\pi}{180}d.$$

0.7.2 Trigonometric Functions

$\sin\theta = \frac{y}{r}$, $\cos\theta = \frac{x}{r}$, $\tan\theta = \frac{y}{x}$

$\csc\theta = \frac{r}{y}$, $\sec\theta = \frac{r}{x}$, $\cot\theta = \frac{x}{y}$



circle of radius r units

$\sin(\theta + 2k\pi) = \sin\theta$, $\cos(\theta + 2k\pi) = \cos\theta$, $\tan(\theta + k\pi) = \tan\theta$ where k is an integer.

0.7.3 Trigonometric Identities and Formulas

$\sin\theta\csc\theta = 1, \quad \cos\theta\sec\theta = 1, \quad \tan\theta\cot\theta = 1.$

$\sin^2\theta + \cos^2\theta = 1, \quad 1 + \tan^2\theta = \sec^2\theta, \quad 1 + \cot^2\theta = \csc^2\theta.$

$\sin(\theta + \beta) = \sin\theta\cos\beta + \cos\theta\sin\beta.$

It is necessary that you be familiar with these formulas. Many more can be deduced.

$\sin(\theta - \beta) = \sin\theta\cos\beta - \cos\theta\sin\beta.$

$\tan(\theta + \beta) = \frac{\tan\theta + \tan\beta}{1 - \tan\theta\tan\beta}.$

$\sin\left(\frac{\pi}{2} \pm \theta\right) = \cos\theta, \quad \cos\left(\frac{\pi}{2} \pm \theta\right) = \mp\sin\theta, \quad \tan\left(\frac{\pi}{2} \pm \theta\right) = \mp\cot\theta.$

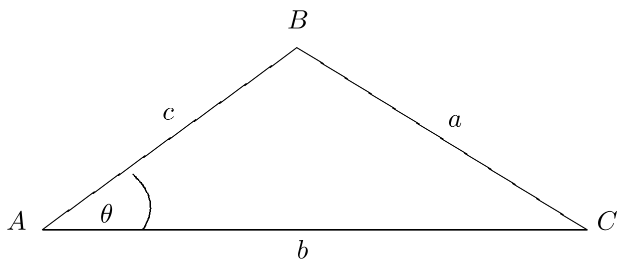
$\cos 2\theta = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta.$

$\sin^2\theta = \frac{1}{2}(1 - \cos 2\theta)$; note: $\sin^2\theta = (\sin\theta)(\sin\theta)$.

$\cos^2\theta = \frac{1}{2}(1 + \cos 2\theta).$

0.7.4 Law of Cosines

Let a, b, c be the lengths of the sides of the triangle ABC. Then $a^2 = b^2 + c^2 - 2bc\cos\theta$.



237. Graph the following functions and state their domain and range.

 (a) $\sin\theta$, (b) $\cos\theta$, and (c) $\tan\theta$.

238. In the following problems calculate the value of the function for the given values of θ.

 (a) $\sin\theta$; where $\theta = 0, 1, 1.5, -2.6, \pi, -\frac{\pi}{2}$, and 5000.

 (b) $\cos\theta$; where $\theta = 0, 1, 2.5, 3, 5280, -782, \pi, \frac{\pi}{2}$, and $\frac{3\pi}{2}$.

 (c) $\tan\theta$; where $\theta = 0, 1, 1.5, -\pi, \frac{\pi}{4}$, and 1000.

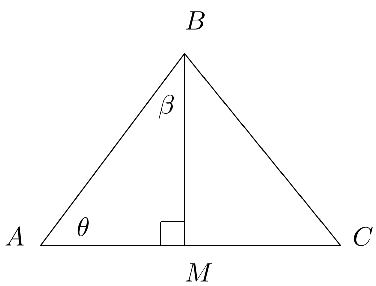
239. Express the following functions in terms of sine and/or cosine functions only.

 (a) $\tan\theta$, (b) $\cos^2\frac{\theta}{2}$, (c) $\sin^2\frac{\theta}{2}$.

240. If the angles of a triangle are x, $x+1$, $x+2$ (in radians), find x.

241. A satellite is launched into a circular orbit about the earth. If the distance of the satellite from the center of the earth is 6,000 miles, how far does it travel while sweeping out an angle of 45° at the center of the earth?

242. Given triangle ABC with $AB = BC = CA = 2$ and $AM = MC$. Find the following:



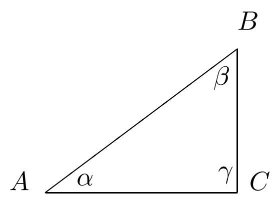
 NOTE: Do not convert your answers to decimal approximations.

 (a) Length of BM.

 (b) θ & β in radians.

 (c) $\sin\theta$, $\cos\theta$, $\sin\beta$, $\cos\beta$, $\tan\theta$ and $\tan\beta$.

243. Given triangle ABC.



If $\gamma = \pi/2$ and $\alpha = \beta$ find β in radians and compute, $\cos\alpha$, $\sin\alpha$, and $\tan\alpha$.

NOTE: Do not convert your answers to decimal approximations.

244. Compute the following values of the functions at the given angle. *Hint:* use trigonometric identities and results of #242 and 243.

(a) $\sin\left(\frac{\pi}{3} + \frac{\pi}{4}\right)$ (b) $\cos\left(\frac{\pi}{3} + \frac{\pi}{4}\right)$

(c) $\cos\left(\frac{\pi}{2} + \pi\right)$ (d) $\sin(3\pi) + \cos(3\pi)$

(e) $\cos\frac{\pi}{4}$ (f) $\sin\left(\frac{\pi}{12}\right)$ $\left[\text{hint:}\frac{\pi}{12} = \frac{\pi}{3} - \frac{\pi}{4}\right]$

(g) $\tan\left(\frac{\pi}{3}\right)$ (h) $\tan\left(\frac{\pi}{6}\right)$

245. At $t = 0$ two cars cross the intersection of two straight roads with velocities v_1 and v_2. The two roads intersect at an angle of θ.

(a) What is the distance between the cars t hours after they have crossed the intersection.
(b) Compute the distance between the cars 1 hour after they have crossed the intersection if
   (i) $v_1 = v_2$ and $\theta = \frac{\pi}{3}$
   (ii) $v_1 = v_2$ and $\theta = \frac{\pi}{4}$
   (iii) $v_1 = v_2$ and $\theta = 0$
   (iv) $v_1 = 2v_2$ and $\theta = \frac{\pi}{3}$

0.8 Lines

In problems 246–254, find the equation of the line through each of the given pairs of points and sketch the graph.

246. $(3, 2), (-2, 4)$
247. $(1, 1), (2, -2)$
248. $(-3, -3), (4, 9)$
249. $(-1, -3), (-2, 5)$
250. $(0, 0), (3, 2)$
251. $(5, 0), (0, 5)$
252. $(-2, -3), (5, -7)$
253. $(\frac{1}{2}, 2), (\frac{5}{2}, -2)$
254. $(\frac{1}{3}, \frac{1}{4}), (\frac{2}{3}, \frac{1}{3})$

255. Find the values of x for which the slope of the segment joining the two given points:

(i) is zero
(ii) does not exist

(iii) is positive

(iv) is negative

(a) $(2,3), (x,5)$ (b) $(6,-1), (3,x)$

(c) $(4,x), (x,2)$ (d) $(-6,x^2), (x^2,-2)$

256. Determine whether the line passing through the first two points is parallel to the line passing through the last two points.

(a) $(6,-2), (0,2);\ (5,1), (12,10)$

(b) $(-2,-4), (-4,-1);\ (7,4), (-3,19)$

(c) $(6,-1), (11,1);\ (5,-2), (20,4)$

(d) $(-1,4), (7,1);\ (4,2), (15,-2)$

In problems 257–264, find the equation of the line with the given slope m, and containing the given point. Sketch the graph.

257. $(\frac{1}{3},3),\ m=\frac{1}{2}$

258. $(-2,5),\ m=-\frac{2}{3}$

259. $(-4,-3),\ m=1$

260. $(-3,-3),\ m=-1$

261. $(0,3),\ m=-2$

262. $(3,0),\ m=2$

263. $(-4,3),\ m=0$

264. $(1,-3),\ m=0$

In problems 265–268, determine the slope, y-intercept, and x-intercept of the graph of the function defined.

265. $\{(x,y) \mid 3x+4y-6=0\}$

266. $\{(x,y) \mid x-2y+4=0\}$

267. $\{(x,y) \mid -4x+5y+12=0\}$

268. $\{(x,y) \mid 2x+y=0\}$

269. (a) What is the equation of the line that passes through the points $(0,3)$ and $(5,0)$.

(b) Show that the equation of the line that passes through the points $(0,b)$ and $(a,0)$ can be written in the form (called the intercept form):

$$\frac{x}{a}+\frac{y}{b}=1 \qquad (a\cdot b\neq 0).$$

270. The graph of a linear function, f, has slope $m=2$. If $(-1,3)$ and $(c,-2)$ belong to the graph, find the number c.

In the following problems find the point of intersection of the two lines if it exists and draw a graph for each problem.

271. $3x+y-1=0$ and $2x+y-1=0$.

272. $-2x+3y-6=0$ and $-2x+3y+3=0$.

273. $-2x+5y+30=0$ and $5x+2y-2=0$.

274. $-x+y-2=0$ and $x+y-2=0$.

275. $y-3x=0$ and $y-3x+1=0$.

276. $y+x+1=0$ and $2y+2x+1=0$.