Contents

Chapter 1

Functions

1.1 Definition of a Function

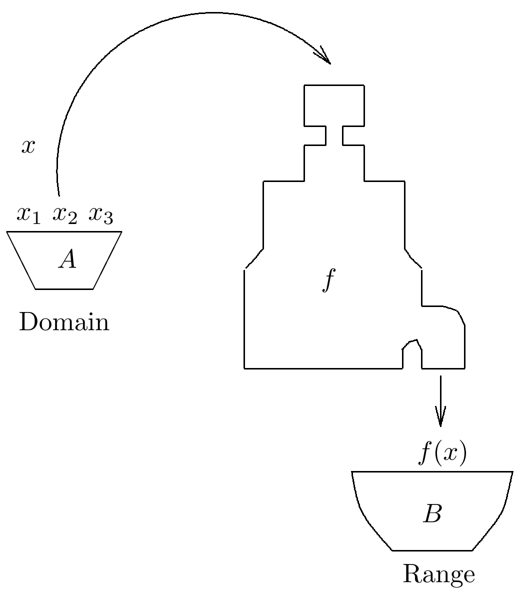
A *function*, f, with domain D, is a *rule* which assigns to each element $x \in D$ a *single* real number, $f(x)$.

The *domain* is usually a set of real numbers.

The *range* of f consists of all numbers of the form $f(x)$, with $x \in D$.

The rule, f, can be thought of as a machine designed to take a specified set of real numbers (the domain) and produce, for each acceptable x of the input, a single real number $f(x)$.

$f : D \to B$



1.2 Examples of Functions

1.2.1 Polynomial Functions

(i) **Constant Functions:** All functions of the form $f : \mathbb{R} \to \{a\}$ where $\mathbb{R}$ is the set of real numbers and $\{a\}$ is the set containing the fixed real number a.

Examples:

(1) $f(x) = 2$. Here f assigns to each number $x \in \mathbb{R}$ the real number 2.

(2) $Z(x) = 0$. To every real number Z assigns the real number zero.
This function is called the zero function, or zero polynomial.

(ii) **Linear Functions:** All functions $L : \mathbb{R} \to \mathbb{R}$ defined by the rule $L(x) = ax + b$, where a and b are any fixed real numbers.

Examples:

(1) $I(x) = x$. I assigns every real number to itself. This function is called the identity function.

(2) $L(x) = -2x$ and $L(x) = 6x - 5$.

(iii) **Polynomial Functions:** Any function of the form $f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_0$ where n is a positive integer or zero and the a_i's are fixed real numbers. Note if $n = 1$ then $f(x) = a_1 x + a_0$ is a linear function and if $n = 0$ then $f(x) = a_0 x^0 = a_0$ is a constant function. For any value, n, this is called a polynomial function of *degree* at most n. The degree equals n iff $a_n \neq 0$.

Examples:

(1) $f(x) = x^2 + 3x + 1$. (2) $f(x) = x^5$.

(3) $f(x) = x^3 + x + 1$. (4) $f(x) = x$.

1.2.2 Rational Functions

A rational function is any function of the type $h(x) = f(x)/g(x)$ where f and g are polynomial functions and g is not the zero polynomial. The *domain* of $h(x)$ is the set of all x such that $g(x) \neq 0$.

Examples:

(1) $h(x) = x^2$. (2) $h(x) = \frac{x}{1-x^2}$.

(3) $h(x) = \frac{x^2}{x}$. (4) $h(x) = \frac{1}{x}$.

1.2.3 Trigonometric Functions

These functions are described on page 16 of Chapter 0 in terms of radian measure. Since angles in radian measure are real numbers the domain of sine and cosine is $\mathbb{R}$. The domains of the other trigonometric functions can be found by expressing them in terms of sine and cosine.

Example: $\tan x = \frac{\sin x}{\cos x}$. Hence the domain of $\tan x$ is all $x \in \mathbb{R}$ such that $\cos x \neq 0$.

1.2.4 The Absolute Value Function

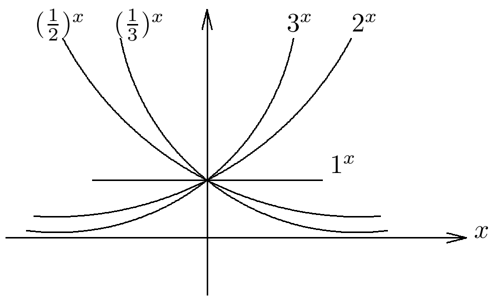
$A(x) = |x|$, where $|x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0. \end{cases}$

1.2.5 The Square Root Function

$f(x) = \sqrt{x}$, where the symbol $\sqrt{x}$ means the positive square root of x and where the domain of this function is $x \geq 0$. The range of this function is $\mathbb{R} \geq 0$.

1.2.6 Exponential Functions

$f(x) = a^x$, where a is any fixed positive number and x is any real number.



Graphs of $f(x) = (1/2)^x$, $f(x) = (1/3)^x$, $f(x) = 3^x$, $f(x) = 2^x$ and $f(x) = 1^x$.

NOTES:

(1) For any choice of $a > 0$, $a^0 = 1$.

(2) For any choice of $a > 0$, $a^x > 0$ for every $x \in \mathbb{R}$.

(3) If $a > 1$ then $1 < a^x < \infty$ for $x > 0$ and $0 < a^x < \infty$ for $x < 0$.

(4) If $a < 1$ then $0 < a^x < 1$ for $x > 0$ and $1 < a^x < \infty$ for $x < 0$.

(5) The laws of exponents stated on page 14 Chapter 0, also apply when α, β are real numbers.

1.2.7 The Logarithmic Function

Let $y = f(x)$ then $f(x) = \log_a x$ iff $x = a^y$ where $x > 0$, $a > 0$ and $a \neq 1$. The number a is fixed. It is called the base of the logarithm.

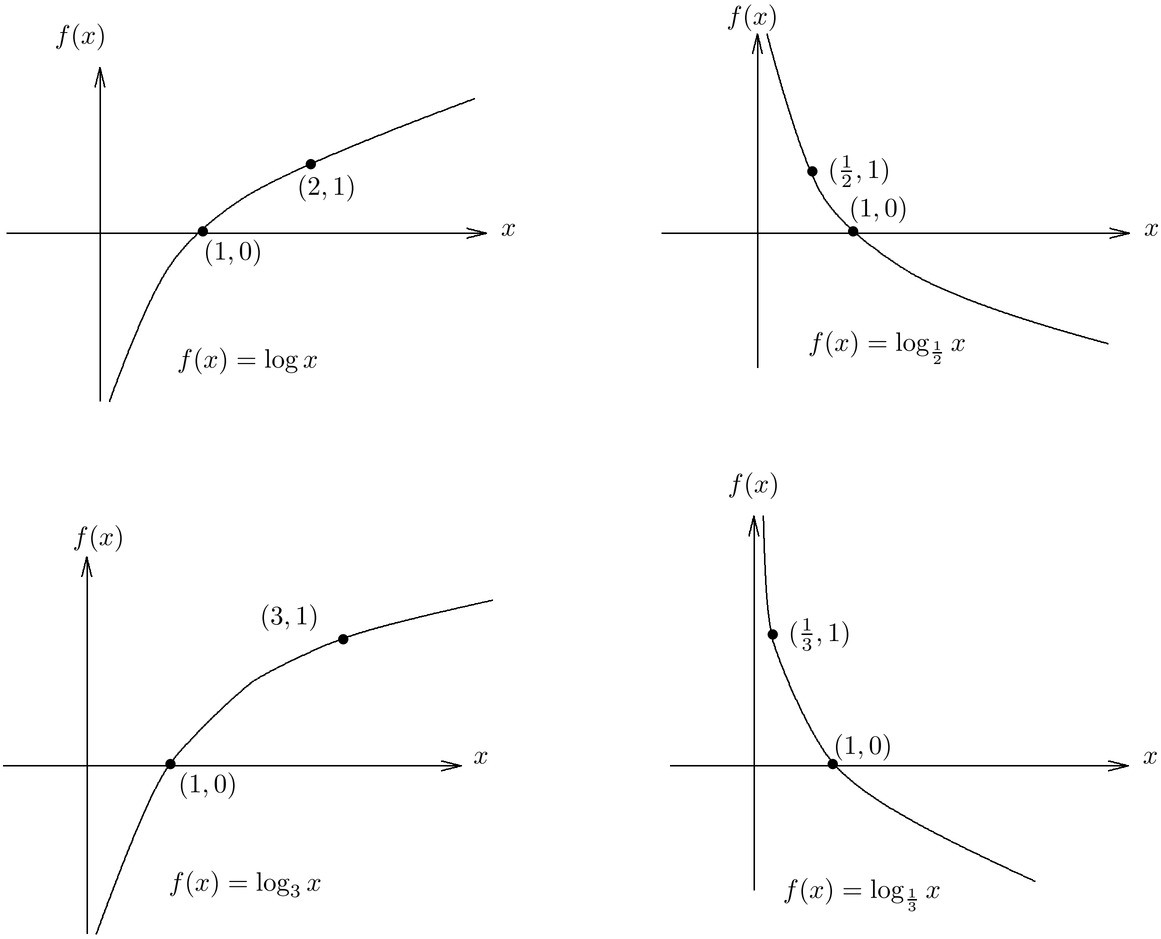
The rule for defining $\log_a x$ in words is:

The logarithm, base a $(a > 0,\ a \neq 1)$, of the number x $(x > 0)$ is the number y such that $a^y = x$.

Domain of $\log_a x$ is $x > 0$.

Range of $\log_a x$ is $\mathbb{R}$.

<u>GRAPHS</u>



Properties of the Logarithmic Functions

If x and z are positive real numbers and p is a real number, then for any base a, $a > 0$ and $a \neq 1$ we have the following properties:

(1) Let $g(x) = a^x$ and $f(z) = \log_a z$ then

  (a) $g\big(f(z)\big) = a^{\log_a z} = z$.

  (b) $f\big(g(x)\big) = \log_a a^x = x$.

(2) Let $f(x) = \log_a x$, then

  $f(1) = 0$ and $f(a) = 1$.

(3) $\log_a(xz) = \log_a x + \log_a z$.

(4) $\log_a(x/z) = \log_a x - \log_a z$.

(5) $\log_a x^p = p \log_a x$.

(6) $\log_a x = \dfrac{\log_b x}{\log_b a}$.

(7) If $a > 1$ then $\log_a x < \log_a z$ if and only if $x < z$.

1.2.8 Floor Function $\lfloor x \rfloor$

Definition:

For every real number x, the value of $\lfloor x \rfloor$ is the greatest integer which is less than or equal to x.

1.2.9 Ceiling Function $\lceil x \rceil$

Definition:

For every real number x the value of $\lceil x \rceil$ is the smallest integer that is greater than or equal to x.

1.2.10 One to One Functions

Notation. 1-1 or *injective*.

Definition. Given a function $f, f : A \to B$, then f is 1-1 or *injective* $\iff$ [if $f(x) = f(z)$ then $x = z$ for every $x, z \in A$].

Examples

1. $f(x) = ax + b$ for $a \neq 0$.
2. $f(x) = x^3$
3. $\sin : \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \to [-1, 1]$
4. $\ell(x) = \log_a x$
5. $E(x) = a^x$
6. $f(x) = \frac{1}{x}$

These functions are *not* 1-1.

1. $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$
2. $\sin : \mathbb{R} \to \mathbb{R}$
3. $|x| : \mathbb{R} \to \mathbb{R}$

1.2.11 Even and Odd Functions

Definitions:

1. Even. A function f is said to be an *even function* if $f(-x) = f(x)$ for all x in the domain of f. It is assumed that for every x in the domain of f, $(-x)$ is also in the domain.

Examples

(a) $\cos : \mathbb{R} \to \mathbb{R}$

(b) $f(x) = x^2$

(c) $f(x) = e^{-x^2}$

(d) $f(x) = |x|$

2. Odd. f is an *odd function* if $f(-x) = -f(x)$ for all x in the domain of f. It is assumed that for every x in the domain of f, $(-x)$ is also in the domain.
Examples

(a) $\sin : \mathbb{R} \to \mathbb{R}$

(b) $f(x) = x^3$

(c) $f(x) = \frac{1}{x}$

(d) $f(x) = xe^{x^2}$

Note.

- Most functions are neither even nor odd. For example, $f(x) = x^2 - x$ is neither even nor odd.
- A polynomial $p(x) = a_n x^n + \cdots + a_0$ is *even* iff all $a_{2i+1} = 0$, i.e. only even powers of x actually appear in $p(x)$.
- A polynomial $p(x) = a_n x^n + \cdots + a_0$ is *odd* iff all $a_{2i} = 0$, i.e. only odd powers of x actually appear in $p(x)$.
- Contrary to the rules of arithmetic:
  (i) the *sum* of (two or more) odd functions is odd.
  (ii) the *product* of an even and an odd function is odd.

In problems 1 to 24 find the following:

(a) Test whether the given function is even or odd.

(b) Domain and range of the given function. Use interval notation where applicable.

(c) For what values of x the function is zero.

(d) For what values of x the function is greater than zero.

(e) For what values of x the function is less than zero.

1. (a) $f(x) = x^3$ (b) $f(x) = x^2,\ x \le 0$
   (c) $f(x) = x^3,\ 1 \le x < 2$ (d) $f(x) = x^2 - x,\ 0 \le x \le 1$
2. (a) $g(x) = x^{\frac{1}{3}}$ (b) $g(x) = -\sqrt{x}$
   (c) $g(x) = \sqrt{x-3}$ (d) $g(x) = \sqrt{x+2}$
3. (a) $f(x) = \frac{x^2}{x}$ (b) $f(x) = \frac{|x|}{x}$
   (c) $f(x) = \frac{x}{|x|}$ (d) $f(x) = \begin{cases} x, & \text{if } x \ge 0 \\ -x, & \text{if } x < 0 \end{cases}$

4. (a) $f(x) = x + |x|$ (b) $f(x) = x^2 + |x|$
   (c) $f(x) = 1 - |x + 1|$ (d) $f(x) = |x^2 - x|$
5. (a) $f(x) = x^2 - 2x + 1$ (b) $f(x) = x^2 + x + 1$
   (c) $f(x) = -x^2 + x - 1$ (d) $f(x) = 2x^2 - 4$
6. (a) $f(x) = \frac{1}{x},\ x < 0$ (b) $f(x) = \frac{1}{x^2}$
   (c) $f(x) = \frac{1}{|x|}$ (d) $f(x) = \frac{x^2-1}{x+1}$
7. (a) $f(x) = \frac{1}{x+1}$ (b) $f(x) = \frac{1}{x^2+1}$
   (c) $f(x) = \frac{3}{x^2-1}$ (d) $f(x) = \frac{2}{|x|+1}$
8. (a) $f(x) = \frac{x-1}{x+1}$ (b) $f(x) = 1 - \frac{1}{x^2+1}$
   (c) $f(x) = \frac{x^2+2}{x-1}$ (d) $f(x) = \frac{3x-1}{2x+1}$
9. (a) $f(x) = x^2 + \frac{1}{x}$ (b) $f(x) = \sqrt{x^2}$
   (c) $f(x) = \sqrt{1 - x^2}$ (d) $f(x) = \sqrt{x^2 + 1}$
10. (a) $f(x) = 1/\sqrt{1 - x^2}$ (b) $f(x) = \frac{\sqrt{1-x^2}}{1-x^2}$
    (c) $f(x) = \frac{1-x^2}{\sqrt{1-x^2}}$ (d) $f(x) = \sqrt{(x + 1)^2}$
11. (a) $f(x) = \begin{cases} \frac{1}{x}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$ (b) $f(x) = \begin{cases} x, & \text{if } x < 0 \\ 1, & \text{if } 0 < x \leq 2 \\ x - 1, & \text{if } 2 < x \end{cases}$
    (c) $f(x) = x|x|$ (d) $f(x) = \begin{cases} x^2, & \text{if } x < 0 \\ 1, & \text{if } x = 0 \\ x, & \text{if } 0 < x < 2 \end{cases}$
12. (a) $f(x) = \sin 2x$ (b) $f(x) = 3 \sin x$
    (c) $f(x) = \sin x - 3$ (d) $f(x) = \cos x + 2$
13. (a) $f(x) = \sin x + \cos x$ (b) $f(x) = |\sin x|$
    (c) $f(x) = \csc x$ (d) $f(x) = \sec x$
14. (a) $f(x) = \tan(x + \pi)$ (b) $f(x) = \sin^2 x + \cos^2 x$
    (c) $f(x) = \sqrt{\sin x}$ (d) $f(x) = \sin |x|$
15. (a) $f(x) = x \sin x$ (b) $f(x) = x^2 \sin x$
    (c) $f(x) = \frac{\sin x}{x}$ (d) $f(x) = \cos x + x$
16. (a) $f(x) = 10^x$ (b) $f(x) = 10^{-x}$
    (c) $f(x) = \frac{8^x}{2^x}$ (d) $f(x) = 10^{x+1}$

Do not attempt to find the range in problem 17.

17. (a) $f(x) = (x + 1)2^x$ (b) $f(x) = x2^x + \frac{3}{2^{-x}}$
    (c) $f(x) = 1 - \frac{1}{2^{-x}}$ (d) $f(x) = \frac{e^{x+1}}{x+1}$
18. (a) $f(x) = a^{x^2+2x+1}$ (b) $f(x) = 2^{x^2-1}$
    (c) $f(x) = 10^{x^2-x-2}$ (d) $f(x) = 2^{\sin x+2}$
19. (a) $f(x) = \log_2(x + 1)$ (b) $f(x) = \log_2 |x|$
    (c) $f(x) = \log_{10}(x^2 - x - 2)$ (d) $f(x) = \log_{10}(x + 1) + \log_{10}(x - 1)$

20. (a) $f(x) = \log_{10}(x+1) + \log_{10} x$ (b) $f(x) = \log_{10}(x^2+x)$
(c) $f(x) = \log_{10}\sqrt{x}$ (d) $f(x) = \log(\sin x)$

Do not attempt to find the range in problem 21 and 22.

21. (a) $f(x) = x^2 \log_{10} x$ (b) $f(x) = \frac{x}{\log_{10} x}$
(c) $f(x) = \frac{\log_{10}(x+1)}{x}$ (d) $f(x) = \log_{10}\left(\frac{x+1}{x-1}\right)$

22. (a) $f(x) = \frac{\log_{10} x}{\log_{10}(x+10)}$ (b) $f(x) = \log_{10}\left(\frac{x}{x+10}\right)$

23. If $L(x) = 3x+4$, find:
(a) $L(0)$ (b) $L(-x)$ (c) $L(\pi)$
(d) $(L(x))^2$ (e) $L(x^2)$ (f) $L(L(x))$
(g) $L(x+h)$ (h) $L(L(x)-4)$ (i) $L(2^x)$

24. If $f(x) = x^2+2$ find:
(a) $f(\sqrt{2})$ (b) $f\left(\frac{1}{x}\right), x \neq 0$ (c) $f(\cos x)$
(d) $f(3x+4)$ (e) $1/f(x)$ (f) $f(f(x))$
(g) $f(x+h)$ (h) $f(10^x)$ (i) $f(\sqrt{x-2})$

25. If $f(x) = \frac{x}{x+1}$, find:
(a) $f(0)$ (b) $f(\sqrt{2})$ (c) $f\left(\frac{x}{1-x}\right)$
(d) $f(2+h)$ (e) $f(\sin x)$ (f) $f(\log_2 x)$
(g) $f(f(x))$ (h) $1/f(x)$ (i) $3f(x) + f(x-1)$

26. If $f(x) = \frac{1}{x-1}$, find:
(a) $f(0)$ (b) $f(-1)$ (c) $f(x+1)$
(d) $f\left(\frac{1}{x}\right)$ (e) $f\left(\tan\frac{\pi}{4}\right)$ (f) $f\left(\sin\frac{\pi}{3}\right)$
(g) $f\left(\frac{1}{x}+1\right)$ (h) $f(f(x))$ (i) $f(f(f(x)))$

27. If $L(x) = 2x-5$ and $G(x) = \frac{x+5}{2}$, find:
(a) $(L+G)(x)$ (b) $3L(1)$ (c) $(L-2G)(x^2)$
(d) $(L^2+G)(x)$ (e) $L(G(x))$ (f) $G(L(x))$
(g) $\frac{L}{G}(2^x)$ (h) $L(G(2^x))$ (i) $(L+G)(1+x)$

28. If $f(x) = \sqrt{x+1}$ and $g(x) = \frac{1}{x+1}$, find:
(a) $(f+g)(x)$ (b) $f(g(x))$ (c) $g(f(x))$
(d) $\frac{f}{g}(x^2)$ (e) $f(f(x))$ (f) $g(g(x))$
(g) $(f-g)(0)$ (h) $(f+g)(x+h)$ (i) $(fg)(x)$

29. If $f(x) = \frac{x-1}{x+1}$, find:
(a) $f\left(\cos\frac{\pi}{2}\right)$ (b) $f\left(\frac{1}{x}\right)$ (c) $f(3x+1)$
(d) $f(f(x))$ (e) $f(2^x)$ (f) $f\left(\frac{x+1}{x-1}\right)$
(g) $f^2(x)$ (h) $f(x+1)$ (i) $f(x-1)$

30. If $f(x) = x^2$ and $g(x) = x^2 + 1$, find:

(a)	$g(g(0))$	(b)	$g(g(g(0)))$	(c)	$f(g(0))$
(d)	$f(g(2))$	(e)	$f(g(\sin x))$	(f)	$g(f(\sin x))$

31. If $f(x) = x^2$ and $g(x) = x^2 + 1$. Show that:

(a) $g\left(\frac{1}{x}\right) = \frac{g(x)}{f(x)}$ (b) $f(g(x)) = g(f(x)) + 2f(x)$

32. Let $S(x) = \sqrt{x}$ and $H(x) = x + 1$. Show that:

(a) $(S(H(x)))^2 = H(x)$ (b) $(H(S(x)))^2 = H(x) + 2S(x)$

33. If $f(x) = 2^x$, find and simplify:

(a)	$f(0)$	(b)	$f(\sin \pi)$	(c)	$f(x^2 + 1)$
(d)	$f(x^2)f(1)$	(e)	$f(x+3)/f(-x)$	(f)	$f(f(3))$
(g)	$f(\sqrt{x})$	(h)	$(f(x))^2$	(i)	$f(f(x))$

34. If $f(x) = 10^x$, find and simplify:

(a)	$f(x) + f(2 + x)$	(b)	$f(x)f(2 + x)$	(c)	$f(x)/f(2 + x)$
(d)	$f(f(2 + x))$	(e)	$(f(2 + x))^2$	(f)	$f((2 + x)^2)$
(g)	$\frac{1}{f(x)}$	(h)	$f(\sin^2 x)f(\cos^2 x)$	(i)	$\frac{f(\cos^2 x)}{f(\sin^2 y)}$

35. If $P(t) = \alpha e^{\beta t}$ where α and β are constants and $e > 1$ is a constant find and simplify:

(a)	$P(0)$	(b)	$P(1)$	(c)	$P(t + 1)$
(d)	$P(t)P(1)$	(e)	$(P(t))^{\frac{1}{\beta}}$	(f)	$P\left(\frac{1}{\beta}\right)$
(g)	$\frac{P(t+1)}{P(1)}$	(h)	$\frac{P(t+1)}{P(t)}$	(i)	$(P(t))^2$

36. If $g(x) = x2^x$ find and simplify:

(a)	$g(0)$	(b)	$g(1)$	(c)	$g(-1)$
(d)	$g(x + 1)$	(e)	$g(x)g(1)$	(f)	$g(x + 1)g(x)$
(g)	$g(g(x))$	(h)	$(g(x))^2$	(i)	$\frac{g(x+1)}{g(x)}$

37. If $C(x) = ½(e^x + e^{-x})$ and $S(x) = ½(e^x - e^{-x})$ where $e > 1$ is a constant, find the following:

(a)	$C(0)$	(b)	$C(1)$	(c)	$S(0)$
(d)	$S(1)$	(e)	$C(\log_e x)$	(f)	$S(\log_e x)$
(g)	$\frac{S(x)}{C(x)}$	(h)	$C(-x)$	(i)	$S(-x)$

38. If $f(x) = \log_2 x$ and $g(x) = 2^x$, find and simplify:

(a)	$f(1)$	(b)	$f(2)$	(c)	$f(x) - f(x + 1)$
(d)	$f(x) + f(2)$	(e)	$f(g(x))$	(f)	$f(f(g(x)))$
(g)	$g(f(x))$	(h)	$f(x) + f(1 + x)$	(i)	$g(g(f(x)))$

39. If $f(x) = \log_5 x$ and $g(x) = 5^x$, find and simplify:

(a)	$f(1)$	(b)	$f(5)$	(c)	$g(f(x))$
(d)	$f(x) + f(125)$	(e)	$f(x) - f(x + 1)$	(f)	$f(g(g(x)))$
(g)	$f(g(x))$	(h)	$f(x) + f(x + 5)$	(i)	$g(g(f(x)))$

40. If $f(x) = \log_{10} x$ and $g(x) = 10^x$, find and simplify:
   (a) $f(1)$ (b) $f(10)$ (c) $g(f(x))$
   (d) $f(3) + f(\sqrt{x})$ (e) $f(x^2 - 1) - f(x + 1)$ (f) $f(f(g(x)))$
   (g) $f(x) + f(10 + x)$ (h) $f(g(x))$ (i) $g(g(f(x)))$

41. If $f(x) = \log_3 x$ and $g(x) = 3^x$, find:
   (a) $f(1)$ (b) $f(3)$ (c) $f(g(x))$
   (d) $f(27) + f(x)$ (e) $f(x^2 - 9) - f(x + 3)$ (f) $f(f(g(x)))$
   (g) $f(x^2 + 27x)$ (h) $g(f(x))$ (i) $g(g(f(x)))$

In problems 42 to 45 find the values of the given expressions.

42. (a) $\log_3 81$ (b) $\log_4 16$ (c) $\log_2 16$
43. (a) $\log_2 \left(\frac{1}{32}\right)$ (b) $\log_3 \left(\frac{1}{27}\right)$ (c) $\log_4 \left(\frac{1}{64}\right)$
44. (a) $\log_2 1$ (b) $\log_7 \left(\frac{1}{49}\right)$ (c) $\log_{13} 13$
45. (a) $\log_{1/2}(8)$ (b) $\log_{1/6} 216$ (c) $\log_{1/4} \left(\frac{1}{16}\right)$

46. With the help of a table of logarithms and the relation $\log_a x = \frac{\log_b x}{\log_b a}$ construct a table of logarithms for the first ten integers for the following bases:
   (a) base 2 (b) base 3 (c) base 5

In problems 47 to 50 the symbol $\log x$ will stand for $\log_a x$. Simplify the given expressions.

47. (a) $\log a^{-x}$ (b) $a^{-\log x}$ (c) $a^{x+\log x}$
48. (a) $\log(xa^{2x})$ (b) $a^{-\log x^2}$ (c) $a^{\log a^x}$
49. (a) $\log(a^{\log a})$ (b) $a^{2\log 3}$ (c) $\log(x^2 a^x)$
50. (a) $\log(a^{x^2-2x})$ (b) $a^{\log(a^x)}$ (c) $a^{2\log x}$

51. Solve for x:
   (a) $\log_5 x = 3$ (b) $\log_6 x = 3$ (c) $\log_2 x = 10$
   (d) $\log_{10} x = 1/2$ (e) $\log_{10} x = 1$ (f) $\log_{16} x = 1/4$

52. Solve for a:
   (a) $\log_a 216 = 3$ (b) $\log_a 625 = 4$ (c) $\log_a \sqrt{a} = 1/2$
   (d) $\log_a \frac{1}{49} = -2$ (e) $\log_a 2 = 1/4$ (f) $\log_a 125 = 3$

53. Solve for y:
   (a) $2^{\log_2 y} = 13$ (b) $6^{\log_6 y} = 21$ (c) $4^{\log_4 y} = 9$
   (d) $y^{\log_4 6} = 6$ (e) $y^{\log_7 14} = 14$ (f) $y^{\log_3 2} = 2$

54. Solve for x:
   (a) $5^{\log_5 7} = x$ (b) $3^{\log_x 5} = 5$ (c) $10^{\log_x 7} = 7$
   (d) $k^{\log_k 4} = x$ (e) $7^{\log_x k} = k$ (f) $8^{\log_8 x} = y$

55. Solve for x:
    (a) $4^x = 7$ (b) $5^{x+1} = 9$ (c) $6^{2x+3} = 354$
    (d) $x^5 = 873$ (e) $x^4 = 687$ (f) $x^{7/2} = 51.4$

56. Solve for x:
    (a) $3^x = 6^{x+3}$ (b) $7^x = 4^{2x-1}$ (c) $2^{x-1} = 5^{2x+1}$
    (d) $8^{x+2} = 3^{3x-1}$ (e) $y = 2^{3x}$ (f) $10y = 10^x$

57. Solve for x: (Note: here $\log x$ means $\log_{10} x$.)
    (a) $\log(3x-1) - \log(x+2) = 2$ (b) $\log(x-\sqrt{6}) + \log(x+\sqrt{6}) = 1$
    (c) $\log(x^2-1) - \log(x+1) = 1$ (d) $\log(x^2-4) - 2\log(x-2) = 2$

58. Solve the following inequalities for x:
    (a) $4^x > \frac{2}{5}$ (b) $3^x < \frac{4}{7}$
    (c) $2^{3x} < 6$ (d) $5^{2x} > 8$

In problems 59 to 69 find functions f and g such that the given function h can be expressed in the form $h = f \circ g$. Neither f nor g should be the identity function.

59. (a) $h(x) = (x+1)^2$ (b) $h(x) = (x-3)^2 + x - 3$
    (c) $h(x) = (x^2-1)^{1/2}$ (d) $h(x) = x^2 - 2x + 1$

60. (a) $h(x) = (x+1)^3 + 3$ (b) $h(x) = \sqrt{x-3}$
    (c) $h(x) = 3\sqrt{x} + 4$ (d) $h(x) = (x-1)^2 + x - 2$

61. (a) $h(x) = (x+1)^2 + x + 2$ (b) $h(x) = 1/\sqrt{x-2}$
    (c) $h(x) = \sqrt{x^3+1}$ (d) $h(x) = 1/(x^2+1)$

62. (a) $h(x) = \sqrt{x^2}$ (b) $h(x) = |x^2+1|$
    (c) $h(x) = |2x+1|$ (d) $h(x) = \sqrt{|x|}$

63. (a) $h(x) = \sin 2x$ (b) $h(x) = \sin x^2$
    (c) $h(x) = \sin^2 x$ (d) $h(x) = \sin(\cos x)$

64. (a) $h(x) = \sin^2 3x$ (b) $h(x) = |\sin x|$
    (c) $h(x) = \cos |x|$ (d) $h(x) = \tan(x^2+1)$

65. (a) $h(x) = \sqrt{\sin x}$ (b) $h(x) = \tan^2 x - 2\tan x + 1$
    (c) $h(x) = 3\sin^2 x + \sin x + 1$ (d) $h(x) = \sin(\cos^2 x)$

66. (a) $h(x) = 2^{-x}$ (b) $h(x) = 2^{x^2}$
    (c) $h(x) = 10^{\sin x}$ (d) $h(x) = 10^{|x|}$

67. (a) $\log_{10}(x^2+1)$ (b) $\log_{10}(\sin x)$
    (c) $h(x) = \log_2 |x|$ (c) $h(x) = \sin(\log_2 x)$

68. (a) $h(x) = e^{2x} + e^x + 1$ (b) $h(x) = \log_2 4x$
    (c) $h(x) = 3\log_{10} x - 2$ (d) $h(x) = (\log_{10} x)^2 - 1$

69. (a) $h(x) = \log_{10} x^2$ (b) $h(x) = (\log_{10} x)^2$
    (c) $h(x) = x^4 + x^2 - 2$ (d) $h(x) = 2^{2x} + e^{x+1} + 1$

In problems 70 to 75 evaluate $\frac{f(x+h)-f(x)}{h}$ $(h \neq 0)$ and $\frac{f(x)-f(a)}{x-a}$ for the given function and given values of a.

70. (a) $f(x) = x^2 + 1,\ a = 0$ and $a = 1$ (b) $f(x) = x^3,\ a = 0$ and $a = -3$

71. (a) $f(x) = x + 3,\ a = -1$ and $a = 2$ (b) $f(x) = \frac{1}{x},\ a = 1$ and $a = 3$

72. (a) $f(x) = 2x + 3$ (b) $f(x) = x^2 + 1$

73. (a) $f(x) = \sqrt{x},\ a = 1$ and $a = 4$ (b) $f(x) = |x|,\ a = 0$ and $a = 4$

74. (a) $f(x) = 2^x,\ a = 0$ and $a = 1$ (b) $f(x) = 10^x,\ a = 0$ and $a = 1$

75. (a) $f(x) = \log_{10} x,\ a = 1$ and $a = 10$ (b) $f(x) = \log_2 x,\ a = 1$ and $a = 2$

76. Given $f(x) = \begin{cases} x & \text{if } x \neq 1 \\ 2 & \text{if } x = 1 \end{cases}$ find $\frac{f(1+h)-f(1)}{h}$ $(h \neq 0)$

77. Given $f(x) = \begin{cases} \sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$ find $\frac{f(h)-f(0)}{h}$ $(h \neq 0)$

78. Given $f(x) = \begin{cases} x & \text{if } x \geq 0 \\ 1 & \text{if } x < 0 \end{cases}$ find $\frac{f(h)-f(0)}{h}$ $(h \neq 0)$

1.3 Inverse Trigonometric Functions

Definitions:

The *inverse sine function*, denoted $\sin^{-1}$ or arcsin, is defined by

$$\sin^{-1} x = y \quad \text{if and only if} \quad \sin y = x$$

where $-1 \leq x \leq 1$ and $-\pi/2 \leq y \leq \pi/2$.

The *inverse cosine function*, denoted by $\cos^{-1}$ or arccos, is defined by

$$\cos^{-1} x = y \quad \text{if and only if} \quad \cos y = x$$

where $-1 \leq x \leq 1$ and $0 \leq y \leq \pi$.

The *inverse tangent* or *arctangent function*, denoted by $\tan^{-1}$ or arctan, is defined by

$$\tan^{-1} x = \arctan x = y \quad \text{if and only if} \quad \tan y = x$$

where x is any real number and $-\pi/2 < y < \pi/2$.

The *inverse secant* or *arcsecant function*, denoted by $\sec^{-1}$ or arcsec, is defined by

$$\sec^{-1} x = \operatorname{arcsec} x = y \quad \text{if and only if} \quad \sec y = x$$

where $|x| \geq 1$ and y is in $[0, \pi/2]$ or in $[\pi, 3\pi/2]$.

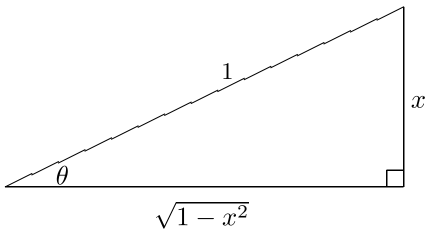
Example: Compute $\cos(\sin^{-1} x)$

Solution:

(a) Let $\theta = \sin^{-1} x$ with the restrictions:

$$-\frac{\pi}{2} \le \theta \le \frac{\pi}{2} \qquad \text{and} \qquad -1 \le x \le 1.$$

(b) Since $\sin\theta = x$ by definition of $\sin^{-1}$, we can construct the given right angle triangle and compute the third side by Pythagoras theorem.



(c) Now $\cos(\sin^{-1} x)$ is $\cos\theta$, so we read the answer off the diagram:

$$\cos(\sin^{-1} x) = \sqrt{1-x^2}.$$

Evaluate

79. (a) $\sin^{-1}\frac{1}{2}$ (b) $\cos^{-1}\frac{1}{2}$ (c) $\tan^{-1} 1$ (d) $\tan^{-1}\sqrt{3}$
80. (a) $\sin^{-1}\frac{1}{\sqrt{2}}$ (b) $\cos^{-1}\sqrt{3/2}$ (c) $\tan^{-1} 0$ (d) $\sin^{-1} 1$
81. (a) $\sin^{-1} 0$ (b) $\cos^{-1} 1$ (c) $\cos^{-1} 0$ (d) $\cot^{-1}(-1)$
82. (a) $\tan^{-1}(-1)$ (b) $\cot^{-1}\sqrt{3}$ (c) $\sin^{-1}(-\frac{1}{2})$ (d) $\cos^{-1}\frac{1}{\sqrt{2}}$
83. $\sin(\sin^{-1} x)$ and $\tan^{-1}(\tan\theta)$
84. $\cos(\sin^{-1} x)$ and $\sin(\cos^{-1} y)$
85. $\cos(\tan^{-1} x)$ and $\cos(\sec^{-1} x)$
86. $\tan(\cos^{-1} x)$ and $\sin(\cos^{-1} 1)$
87. $\cos(\sin^{-1}\frac{1}{2})$ and $\tan(\cos^{-1} 0)$

In problems 88 to 123 find the derivative.

88. $f(x) = \sin^{-1}(x/a),\ a > 0$
89. $f(x) = \cos^{-1}(x/a),\ a > 0$
90. $f(x) = \tan^{-1}(x/a),\ a \neq 0$
91. $f(x) = \cos^{-1}(ax)$
92. $f(x) = \tan^{-1}(ax)$
93. $f(x) = \cos^{-1}(1/x)$
94. $f(x) = \sin^{-1}\sqrt{x}$
95. $u(t) = (1+t^2)\tan^{-1} t$
96. $g(t) = \cos^{-1}(2-3t)$
97. $h(x) = \sin^{-1}(ax+b)$
98. $f(t) = \sin^{-1}\left(\frac{t-1}{2}\right)$
99. $g(x) = x\cos^{-1} x$
100. $h(x) = \frac{\cot(3x)}{1+x^2}$
101. $f(x) = \sin^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right)$
102. $f(x) = \tan^{-1}[(x-3)^2]$
103. $f(x) = \cot^{-1}[(1-x)/(1+x)]$

104. $g(x) = e^x \sin^{-1} x$

105. $g(t) = \cos^{-1}(e^t)$

106. $g(t) = e^{\tan^{-1} x}$

107. $h(x) = e^{\cos^{-1} x}$

108. $f(t) = \frac{\tan^{-1} t}{t}$

109. $f(x) = \ln\left(\tan^{-1} x\right)$

110. $g(x) = \ln(\sin^{-1} x)$

111. $h(x) = \ln\left(e^{\tan^{-1} x}\right)$

112. $h(x) = \ln\left(e^{\sin^{-1} x}\right)$

113. $f(x) = \ln\left(e^{\sin^{-1} x}\right)$

114. $g(x) = x \ln\left(e^{\sin^{-1} x}\right)$

115. $f(x) = \sin^{-1}(\ln x)$

116. $f(x) = \tan^{-1}(\ln x)$

117. $F(x) = X^{\sin^{-1} x}$

118. $G(x) = [\sin x]^{\sin^{-1} x}$

119. $G(x) = [\sin^{-1} x]^x$

120. $F(x) = \sin(\sin^{-1}(\sin x))$

121. $H(x) = \cos^{-1}(\cos x)$

122. Let $f(x) = \sin^{-1} x + \cos^{-1} x$
   - (a) Find $f(1)$
   - (b) Find $f'(x)$
   - (c) What can you conclude about $f(x)$?
   - (d) Use result (c) to evaluate $f\left(\frac{1}{3}\right)$, $f\left(\frac{1}{16}\right)$ and $f\left(\frac{3}{4}\right)$

123. Let $f(x) = \sec^{-1} x + \csc^{-1} x$
   - (a) Find $f(1)$
   - (b) Find $f'(x)$
   - (c) What can you conclude about $f(x)$?