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Chapter 11

Techniques of Integration

11.1 Basic Substitution and Formulas

Definition

The *indefinite integral* of a function f , written

$$\int f(x)dx,$$

is the most general antiderivative of f ; that is,

$$\int f(x)dx = F(x) + C \quad (C \text{ is any constant})$$

if and only if $F'(x) = f(x)$.

Basic Formulas: (Memorize this list)

$$\int u^n du = \frac{u^{n+1}}{n+1} + C \quad (n \neq -1)$$

$$\int \frac{du}{u} = \ln |u| + C, \quad \text{if } u \neq 0$$

$$\int e^u du = e^u + C$$

$$\int a^u du = \frac{a^u}{\ln a} + C$$

$$\int \sin u du = -\cos u + C$$

$$\int \cos u du = \sin u + C$$

$$\int \sec^2 u du = \tan u + C$$

$$\int \csc^2 u du = -\cot u + C$$

$$\int \sec u \tan u du = \sec u + C$$

$$\int \csc u \cot u du = -\csc u + C$$

$$\int \frac{du}{\sqrt{1-u^2}} = \sin^{-1} u + C$$

$$\int \frac{du}{1+u^2} = \tan^{-1} u + C$$

$$\int \frac{du}{|u|\sqrt{u^2-1}} = \sec^{-1} u + C, \quad |u| > 1$$

$$\int \tan u \, du = -\ln |\cos u| + C \quad \text{or} \quad \ln |\sec u| + C$$

$$\int \cot u \, du = \ln |\sin u| + C$$

$$\int \sec u \, du = \ln |\sec u + \tan u| + C$$

$$\int \csc u \, du = \ln |\csc u - \cot u| + C$$

$$\int [f(u) + g(u)] \, du = \int f(u) \, du + \int g(u) \, du$$

$$\int kf(u) \, du = k \int f(u) \, du \quad \text{if } k \text{ is a constant}$$

$$1. \int (x^2 + a)^2 \, dx$$

$$2. \int (x - 3)^2 \, dx$$

$$3. \int x^{\frac{5}{2}} \, dx$$

$$4. \int \frac{a\pi \, dx}{x}$$

$$5. \int [(x+1)^3 + e^x] \, dx$$

$$6. \int \frac{x^2 e^x + x}{x^2} \, dx$$

$$7. \int 10 \sec^2 x \, dx$$

$$8. \int 16(2^x + 1) \, dx$$

$$9. \int (\sin x + \cos x) \, dx$$

$$10. \int (\sqrt{x} + \sec x \tan x) \, dx$$

$$11. \int (\sec^2 x + 1) \, dx$$

$$12. \int \left(x^2 + \frac{3x^{\frac{2}{3}}}{2} + \frac{1}{x^2} \right) \, dx$$

$$13. \int \left(5x - \frac{4}{x^3} + \frac{3}{x^5} \right) \, dx$$

$$14. \int \sqrt{x} \, dx$$

$$15. \int x\sqrt{x} \, dx$$

$$16. \int \left(x\sqrt{x} - \frac{1}{\sqrt{x}} \right) \, dx$$

$$17. \int \frac{x}{\sqrt{x}} \, dx$$

$$18. \int \frac{x^5 + x^3 + 2}{1 + x^2} \, dx \quad (\text{Hint: divide})$$

$$19. \int \frac{x^3 - a^3}{x - a} \, dx \quad \left[\frac{x^3}{3} + \frac{a}{2}x^2 + a^2 + x + A \right]$$

$$20. \int \frac{(x^3 + 27) \, dx}{x^2 - 3x + 9}$$

$$21. \int \frac{1 - x^2}{1 - x^4} \, dx \quad [\tan^{-1} x + c]$$

$$22. \int \left(x + \frac{1}{\sqrt{1 - x^2}} \right) \, dx$$

$$23. \int \left(\frac{x\sqrt{1 - x^2} + 4}{\sqrt{1 - x^2}} \right) \, dx$$

$$24. \int \left(\frac{x^3 - 8}{x^2 + 2x + 4} \right) \, dx$$

$$25. \int \frac{x^3 + x^2 - x - 1}{x - 1} \, dx$$

$$26. \int \frac{d\theta}{2\theta}$$

27. $\int \sqrt{2px} \, dx$

29. $\int (1+x^2)^{\frac{1}{2}} x \, dx$

31. $\int \frac{x \, dx}{(x^2-1)^{\frac{3}{2}}}$

33. $\int \frac{(x+1)dx}{x^2+2x+2}$

35. $\int \frac{4dt}{5t+2}$

37. $\int \frac{(2x-5)dx}{x^2-5x+3}$

39. $\int \frac{x^3}{x-1} \, dx$

41. $\int \frac{x+6}{(x+2)^2} \, dx$ [Hint: $\frac{x+6}{(x+2)^2} = \frac{(x+2)+4}{(x+2)^2}$]

43. $\int \frac{dx}{\sqrt{x}(1+\sqrt{x})}$

45. $\int x\sqrt{x+1}dx$

47. $\int \frac{dt}{1+\sqrt{t}}$

49. $\int \frac{(1-2x)^2}{x} \, dx$

51. $\int \frac{\csc^2 x \, dx}{1+\cot x}$

53. $\int \frac{\cos x}{\sin x} \, dx$

55. $\int \sin \theta \sqrt{\cos \theta} \, d\theta$

57. $\int x \sin x^2 \, dx$

59. $\int \frac{\cos x \, dx}{\sin^4 x}$

61. $\int \frac{\sin t \, dt}{(4+\cos t)^{\frac{3}{2}}}$

28. $\int \frac{d\theta}{\sec \theta}$

30. $\int x^2(2x^3+1)^2 \, dx$

32. $\int \frac{x \, dx}{3x^2+1}$

34. $\int \frac{2dy}{3y-4}$

36. $\int \frac{x^2 dx}{x^3+4}$

38. $\int \frac{v \, dv}{6v^2-1}$

40. $\int \frac{(y-3)dy}{y^2-6y+1}$

42. $\int \frac{y \, dy}{(1+y^2)^4}$

44. $\int \frac{v+3}{v-1} \, dv$

46. $\int \frac{x^3 dx}{(x^4+16)^2}$

48. $\int \frac{(x^2+1)^2}{x^3} \, dx$

50. $\int \frac{\cos \beta \, d\beta}{2+3 \sin \beta}$

52. $\int \frac{\sin 2t \, dt}{4-3 \cos 2t}$

54. $\int \sin^2 \theta \cos \theta \, d\theta$

56. $\int \frac{\sin \sqrt{x}}{\sqrt{x}} \, dx$

58. $\int \frac{\sec^2 2t \, dt}{1+\tan 2t}$

60. $\int \sin 4x \cos 4x \, dx$

62. $\int \sec^5 x \tan x \, dx$

63. $\int \frac{e^{\sqrt{x}}}{\sqrt{x}} dx$

65. $\int \frac{e^y dy}{e^y - 1}$

67. $\int \frac{dx}{e^x + 1}$

69. $\int \frac{dx}{e^{-x} + e^x}$

71. $\int \frac{\sin(\ln x) dx}{x}$

73. $\int \frac{e^{\ln x}}{x} dx$

75. $\int \frac{\ln x dx}{x}$

77. $\int \frac{dx}{x(1 + 2 \ln x)}$

79. $\int \frac{(1 + 2 \ln x)^4}{x} dx$

81. $\int \frac{(\ln y)^{\frac{3}{2}}}{y} dy$

82. $\int \frac{dx}{\sqrt{6x - 4x^2}}$

64. $\int x^3 e^{2x^2} dx$

66. $\int \frac{e^{-x} dx}{1 + e^{-x}}$

68. $\int \frac{e^x + 1}{e^x - 1} dx$

70. $\int e^t \sec e^t \tan e^t dt$

72. $\int e^{\sin \theta} \cos \theta d\theta$

74. $\int e^{\ln x} dx$

76. $\int e^x (4 - e^x)^{\frac{3}{2}} dx$

78. $\int \frac{x^2 \ln^3(1 + x^3)}{1 + x^3} dx$

80. $\int \tan \theta \ln(\cos \theta) d\theta$

Solution to #82: Let $I = \int \frac{dx}{\sqrt{6x - 4x^2}}$

$$6x - 4x^2 = -4 \left(x^2 - \frac{3}{2}x + \frac{9}{16} \right) + \frac{9}{4} = \frac{9}{4} - 4 \left(x - \frac{3}{4} \right)^2$$

$$\therefore I = \int \frac{dx}{\sqrt{\frac{9}{4} - 4 \left(x - \frac{3}{4} \right)^2}}$$

Let $u = 2 \left(x - \frac{3}{4} \right) \therefore du = 2dx$

Now $I = \frac{1}{2} \int \frac{du}{\sqrt{\frac{9}{4} - u^2}} = \frac{1}{2} \sin^{-1} \frac{u}{3/2} + C$

$$\therefore I = \frac{1}{2} \sin^{-1} \frac{4x-3}{3} + C$$

$$83. \int \frac{x dx}{x^2 + x + 1}$$

Solution for #83: Let $H = \int \frac{x dx}{x^2 + x + 1}$

$$x^2 + x + 1 = (x^2 + x + \frac{1}{4}) + \frac{3}{4} = (x + \frac{1}{2})^2 + \frac{3}{4};$$

let $u = x + \frac{1}{2}$; $du = dx$;

$$\text{then } H = \int \frac{(u - \frac{1}{2})du}{u^2 + \frac{3}{4}} = \int \frac{u du}{u^2 + \frac{3}{4}} - \frac{1}{2} \int \frac{du}{u^2 + (\frac{\sqrt{3}}{2})^2}$$

In the first integral let $w = u^2 + \frac{3}{4}$ and we have $2u du = dw$

$$\therefore \int \frac{u du}{u^2 + \frac{3}{4}} = \frac{1}{2} \int \frac{dw}{w} = \frac{1}{2} \ln |w| + C_1$$

$$\text{Also } \int \frac{du}{u^2 + (\frac{\sqrt{3}}{2})^2} = \frac{2}{\sqrt{3}} \tan^{-1} \left(\frac{2u}{\sqrt{3}} \right) + C_2$$

$$\therefore H = \frac{1}{2} \ln |x^2 + x + 1| - \frac{1}{\sqrt{3}} \tan^{-1} \left(\frac{2x+1}{\sqrt{3}} \right) + C$$

$$84. \int \frac{dx}{x^2 + 3x + 5}$$

$$85. \int \frac{dx}{3x^2 - 12x + 16} \quad \left[\frac{\sqrt{3}}{6} \tan^{-1} \frac{\sqrt{3}}{2}(x - 2) + C \right]$$

$$86. \int \frac{dx}{\sqrt{4x - x^2}}$$

$$87. \int \frac{dx}{\sqrt{9x - 4x^2}} \quad \left[\frac{1}{2} \sin^{-1} \frac{8x-9}{9} + C \right]$$

$$88. \int \frac{dx}{12x^2 + 56x + 72}$$

$$89. \int \frac{8dx}{9x^2 - 12x + 20} \quad \left[\frac{2}{3} \tan^{-1} \frac{3x-2}{4} + C \right]$$

$$90. \int \frac{dx}{\sqrt{8x - 25x^2}}$$

$$91. \int \frac{dx}{\sqrt{-x^2 - 2x + 8}} \quad \left[\sin^{-1} \frac{x+1}{3} + C \right]$$

$$92. \int \frac{dx}{2x^2 + 3x + 2}$$

$$93. \int \frac{x dx}{\sqrt{3 + 4x - 4x^2}}$$

$$94. \int \frac{x \, dx}{\sqrt{4x - x^2}} \quad \left[-\sqrt{4x - x^2} + \sin^{-1} \frac{x-2}{2} + C \right]$$

$$95. \int \frac{(3x - 1) \, dx}{9x^2 + 6x + 26}$$

$$96. \int \frac{x \, dx}{\sqrt{6 + x - 2x^2}}$$

$$97. \int \frac{dx}{(3x + 1)\sqrt{3x^2 + 2x - 5}} \quad \left[\frac{1}{4\sqrt{3}} \sec^{-1} \frac{4}{3x+1} + C \right]$$

$$98. \int \frac{dx}{(x + 1)\sqrt{x^2 + 2x - 8}}$$

$$99. \int \sin^{\frac{1}{2}} \theta \cos^3 \theta \, d\theta \quad \left[\frac{2}{3} \sin^{\frac{3}{2}} \theta - \frac{2}{7} \sin^{\frac{7}{2}} \theta + C \right]$$

$$100. \int \cos^{-\frac{3}{2}} \theta \sin \theta \, d\theta$$

$$101. \int \cos^3 x \sin^3 x \, dx$$

$$102. \int \sin \theta \cos \theta \, d\theta \quad \left[\frac{1}{2} \sin^2 \theta + C_1 = -\frac{1}{2} \cos^2 \theta + C_2 = -\frac{1}{4} \cos 2\theta + C \right]$$

$$103. \int \cos^3 3\theta \, d\theta$$

$$104. \int \frac{\sin^3 x}{\cos^2 x} \, dx$$

$$105. \int \cos^3 x \csc^3 x \, dx$$

$$106. \int \sin^3 \frac{x}{2} \, dx$$

$$107. \int \tan^2 x \sec^4 x \, dx$$

$$108. \int \tan^3 \theta \sec^4 \theta \, d\theta \quad \left[\frac{1}{6} \sec^6 \theta - \frac{1}{4} \sec^4 \theta + C \right]$$

$$109. \int \tan^3 \theta \sec^{\frac{3}{2}} \theta \, d\theta$$

$$110. \int (\tan^2 \theta + \tan^4 \theta) \, d\theta$$

$$111. \int \cot^2 \frac{x}{3} \, dx$$

$$112. \int \csc^4 \alpha x \, dx \quad \left[-\frac{1}{\alpha} \cot \alpha x - \frac{1}{3\alpha} \cot^3 \alpha x + C \right]$$

113. $\int \cot x \csc^2 x \, dx$

114. $\int \sin^6 x \, dx$

115. $\int \sin^4 x \cos^2 x \, dx$

116. $\int \tan^2 x \cos^4 x \, dx$

117. $\int \sin^4 x \cos^4 x \, dx$

118. $\int \cos 2x \sin 3x \, dx$

$[\frac{1}{2} \sin 5x + \frac{1}{2} \sin x + C]$

119. $\int \cos 5x \cos 7x \, dx - \int \sin 5x \sin 7x \, dx$

120. $\int \sin^{-\frac{5}{2}} u \cos^3 u \, du$

11.2 Integration by Parts

121. $\int x \cos x \, dx$

122. $\int x \sin 2x \, dx$

123. $\int x^2 \sin 2x \, dx$

124. $\int \frac{x^2 dx}{\sec x}$

125. $\int \tan^{-1} x \, dx$

126. $\int x \tan^{-1} x \, dx$

$[\frac{1}{2}x^2 \tan^{-1} x + \frac{1}{2} \tan^{-1} x - \frac{1}{2}x + C]$

127. $\int \sin^{-1} \left(\frac{x}{a} \right) \, dx, \quad a > 0$

128. $\int \sqrt{1-x^2} \, dx$

129. $\int \ln x \, dx$

130. $\int x \ln x \, dx$

131. $\int x^2 \ln x \, dx$

132. $\int x^{-1} \ln x \, dx$

133. $\int \sec^3 \theta \, d\theta$

134. $\int \cos \sqrt{y} \, dy$

135. $\int \sqrt{y} \cos \sqrt{y} \, dy$ $[2y \sin \sqrt{y} + 4\sqrt{y} \cos \sqrt{y} - 4 \sin \sqrt{y} + C]$

136. $\int \ln^2 x \, dx$

137. $\int e^x \sin x \, dx$

138. $\int e^{-2x} \cos x \, dx$

139. $\int 2^x \sin x \, dx$

140. $\int \frac{\sin^{-1} \sqrt{x} \, dx}{\sqrt{x}}$

141. $\int \tan^{-1} \sqrt{x} \, dx$ $[(x+1) \tan^{-1} \sqrt{x} - \sqrt{x} + C]$

142. $\int x^3 e^{x^2} \, dx$

143. $\int x^3 e^{2x} \, dx$

144. $\int \theta \sec^2 a\theta \, d\theta$

145. $\int x^2 \sin^{-1} x \, dx$ $[\frac{1}{3}x^3 \sin^{-1} x + \frac{1}{9}(2+x^2)\sqrt{1-x^2} + C]$

146. $\int \sin^3 x \, dx$

147. $\int \cos 2x \sin x \, dx$

148. $\int x^2 e^{-x} \, dx$

$$149. \int \sin \theta \ln(\cos \theta) d\theta \quad [\cos \theta(1 - \ln(\cos \theta)) + C]$$

$$150. \int xa^x dx$$

$$151. \int \theta \sin \theta \cos \theta d\theta$$

$$152. \int \frac{\ln(\ln x)}{x} dx$$

$$153. \int \frac{\ln \sqrt{x+a}}{x+a} dx \quad \left[\frac{1}{4} \ln^2(x+a) + C\right]$$

$$154. \int x^3 \sqrt{a^2 - x^2} dx$$

11.3 Improper Integrals

In the following problems determine whether or not each of the improper integrals is convergent, and compute its value if it is.

$$155. \int_0^a \frac{dx}{\sqrt{a^2 - x^2}}$$

$$\text{Solution for \#155: Let } A = \lim_{t \rightarrow a} \int_0^t \frac{dx}{\sqrt{a^2 - x^2}}$$

$$\therefore A = \lim_{x \rightarrow a} (\sin^{-1} \frac{x}{a}) = \frac{\pi}{2}$$

$$156. \int_0^2 \frac{dx}{x^2}$$

$$157. \int_0^2 \frac{dx}{x^2 - 4}$$

$$158. \int_{-2}^0 \frac{dx}{x+2}$$

$$159. \int_0^1 \frac{dx}{\sqrt{x}} \quad [2]$$

$$160. \int_0^1 \frac{dx}{\sqrt{x}} \quad (\text{use substitution } u = \sqrt{x})$$

$$161. \int_0^1 \frac{dx}{x^n}, 0 < n < 1$$

$$162. \int_0^1 \frac{dx}{x^n}, n \geq 1$$

$$163. \int_0^1 \frac{dx}{\sqrt{x(1-x)}} \quad [\pi]$$

$$164. \int_0^a \frac{dx}{(x-a)^2}$$

$$165. \int_2^3 \frac{dx}{\sqrt{x-2}}$$

$$166. \int_0^{\frac{\pi}{2}} \frac{\cos \theta d\theta}{\sqrt{1-\sin \theta}}$$

$$167. \int_1^3 \frac{1}{\sqrt{x-1}} dx \quad [2\sqrt{2}]$$

$$168. \int_0^{\frac{\pi}{2}} \cot \theta d\theta$$

$$169. \int_0^{\frac{\pi}{4}} \frac{\sec^2 x dx}{\sqrt{\tan x}}$$

$$170. \int_{-1}^1 x^{-3} dx$$

$$171. \int_0^{\frac{\pi}{2}} \sec \theta d\theta$$

$$172. \int_{-1}^1 \frac{dx}{x^2}$$

$$173. \int_0^3 \frac{dx}{(x-1)^{\frac{2}{3}}}$$

$$174. \int_0^{\infty} \frac{dx}{1+x^2} \quad \left[\frac{\pi}{2}\right]$$

$$175. \int_0^{\infty} e^{-x} \sin x dx$$

$$176. \int_0^{\infty} \frac{x^2}{1+x^2} dx$$

$$177. \int_0^{\infty} \frac{dx}{x^3}$$

$$178. \int_0^{\infty} \frac{1}{(x+1)^{\frac{3}{2}}} dx$$

$$179. \int_e^{\infty} \frac{dx}{x \ln^2 x} \quad [1]$$

$$180. \int_0^{\infty} t^2 e^{-t} dt$$

181. $\int_{-\infty}^0 x^2 e^x dx$

182. $\int_0^{\infty} \frac{dx}{(x^2 + 1)^2}$

183. $\int_0^{\infty} \frac{e^{-\sqrt{x}}}{\sqrt{x}} dx$

184. $\int_0^{\infty} x e^{-x^2} dx$

[$\frac{1}{2}$]

185. $\int_{-\infty}^{\infty} x^2 e^{-x^3} dx$

186. $\int_{-\infty}^{\infty} x^3 e^{-x^4} dx$

187. $\int_{2a}^{\infty} \frac{dx}{x^2 - a^2}, a > 0$

188. $\int_0^{\infty} \frac{dx}{(x+1)(x+2)}$

189. $\int_{-4}^{\infty} \frac{dx}{x\sqrt{x+4}}$