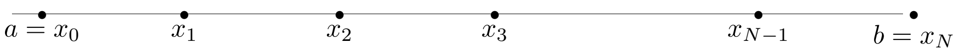
Contents

Chapter 13

Numerical Integration

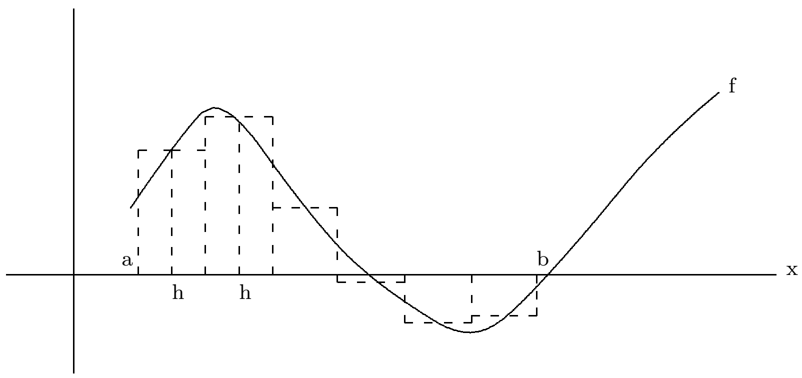
13.1 Midpoint Rule

Divide the interval $[a, b]$ into N equal subintervals as shown in the diagram.



Each subinterval has length $h = \frac{b-a}{N}$. The midpoint of the subinterval $[x_i, x_{i+1}]$ is $m_i = \frac{x_i + x_{i+1}}{2}$.

Now we draw the graph of f



The signed area M_N of the shaded rectangles is

$$M_N = hf(m_0) + hf(m_1) + \ldots + hf(m_{N-1}) = h \sum_{i=0}^{N-1} f(m_i).$$

It is called the midpoint approximation to the definite integral $\int_a^b f(x)dx$.

The *error* is given by the formula

$$\int_a^b f(x)dx = \underbrace{\qquad M_N \qquad}_{\text{approximation}} + \underbrace{\qquad E_N \qquad}_{\text{error}}.$$

If B is a number such that $|f''(z)| \leq B$ for all $z \in [a, b]$ then

$$|E_N| \leq \frac{(b-a)^3 B}{24N^2}.$$

Note: The approximation and its accuracy depends on the number, N, of subintervals.

<u>Example 1</u>:

Approximate $I = \displaystyle\int_{\frac{\pi}{4}}^{\pi} \sin x\, dx$ by the midpoint rule with $N = 4$ and estimate the accuracy of this approximation.

Solution:

Step size: $h = \frac{\pi - \frac{\pi}{4}}{4} = \frac{3\pi}{16}$.

Midpoints:
$m_0 = \frac{\pi}{4} + \frac{3\pi}{32} = \frac{11\pi}{32} = 1.0799225$
$m_1 = \frac{\pi}{4} + \frac{9\pi}{32} = \frac{17\pi}{32} = 1.6689711$
$m_2 = \frac{\pi}{4} + \frac{15\pi}{32} = \frac{23\pi}{32} = 2.2580197$
$m_3 = \frac{\pi}{4} + \frac{21\pi}{32} = \frac{29\pi}{32} = 2.8470683.$

Approximation:

$$\begin{aligned} M_4 &= h\{\sin(m_0) + \sin(m_1) + \sin(m_2) + \sin(m_3)\} \\ M_4 &= \tfrac{3\pi}{16}(.88192127 + .99518473 + .77301046 + .29028468) \\ &= \tfrac{3\pi}{16}(2.9404011) = 1.7320392 \end{aligned}$$

Error analysis:

$$f'(x) = \cos x \quad \text{and} \quad f''(x) = -\sin x$$

$$|f''(x)| = |\sin x| \leq 1 \quad \text{for all } x \in \mathbb{R}.$$

The bound on the error is given by

$$|E_4| \leq \frac{(b-a)^3 B}{24N^2}$$

so

$$|E_4| \leq \left(\frac{3\pi}{16}\right)^3 \frac{1}{24 \cdot 16} < .0340645.$$

There $1.7320392 - .0341 < I < 1.7320392 + .0341$.

Observation:

$\int_{\frac{\pi}{4}}^{\pi} \sin x\, dx = -\cos(\pi) + \cos\left(\frac{\pi}{4}\right) \cong 1.7071068$ by the fundamental theorem. Therefore

$$|E_4| \cong .0249324 < .0341,$$

as predicted by our error analysis.

Example 2:

Approximate $L = \int_1^2 \frac{dx}{x}$ by the midpoint rule within $\pm 10^{-3}$.

Solution:

In this problem we have first to decide how many subintervals to use; i.e. we have to decide what N to choose. We use our formula for the error bound to compute N.

Since $|E_N| < \frac{(b-a)^3 B}{24N^2}$ were need $\frac{(2-1)^3 B}{24N^2} < 10^{-3}$ to guarantee $|E_N| < 10^{-3}$.

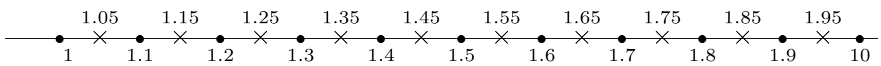
Now $f(x) = \frac{1}{x}$, $f'(x) = -\frac{1}{x^2}$, and $f''(x) = \frac{2}{x^3}$.

And $|f''(z)| = \frac{2}{z^3} \leq 2$ on $[1, 2]$ allows us to pick $B = 2$. Therefore we need $\frac{2}{24N^2} < 10^{-3}$ or

$$\begin{aligned} N^2 &> \tfrac{1\times 10^3}{12} \\ N &> \tfrac{10\sqrt{10}}{\sqrt{12}} \cong 9.129 \end{aligned}$$

Since we need $N > 9.129$ we pick $N = 10$. This gives $h = \frac{2-1}{10} = \frac{1}{10}$.

Midpoints:



$M_{10} = h\left(\frac{1}{m_0} + \frac{1}{m_1} + \ldots + \frac{1}{m_9}\right)$

$M_{10} = \frac{1}{10}(6.9283536) = .69283536.$

Note: $\int_1^2 \frac{dx}{x} = \ln 2 \cong .69314718.$

$\ln 2 - M_{10} \cong .000312 < .001.$

Problems

For the given function f, interval $[a, b]$ and partition N compute M_N, the midpoint approximation of $\int_a^b f(x)dx$, and a bound on the error term.

1. $f(x) = \frac{1}{x}$, $[1.5, 3]$, and $N = 6$

2. $f(x) = \sin x^2, \quad [0, \sqrt{\pi}], \quad \text{and} \quad N = 6$
3. $f(x) = e^{-x^2}, \quad [0, .8], \quad \text{and} \quad N = 8$
4. $f(x) = \frac{1}{1+x^4}, \quad [0, 1], \quad \text{and} \quad N = 4$
5. $f(x) = \sqrt{1 + \cos^2 x}, \quad [\frac{\pi}{2}, \pi], \quad \text{and} \quad N = 6$
6. $f(x) = \begin{cases} \frac{\sin x}{x} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}, \quad [0, \frac{\pi}{2}], \quad \text{and} \quad N = 4$
7. $f(x) = \frac{\sin 2x}{\cos^2 2x}, \quad [0, \frac{\pi}{6}], \quad \text{and} \quad N = 5$
8. $f(x) = \arctan \sqrt{x}, \quad [9, 13], \quad \text{and} \quad N = 4$
9. $f(x) = e^{\sqrt{x}}, \quad [1, 4], \quad \text{and} \quad N = 4$
10. $f(x) = \tan(x^2), \quad [0, \sqrt{\frac{\pi}{4}}], \quad \text{and} \quad N = 4$
11. $f(x) = \sqrt{1 + x^3}, \quad [0, 2], \quad \text{and} \quad N = 2$
12. $f(x) = \frac{1}{\sqrt{1+x^3}}, \quad [0, 2], \quad \text{and} \quad N = 4$
13. $f(x) = \sqrt{x} \sin x, \quad [0, \pi], \quad \text{and} \quad N = 4$
14. $f(x) = x \tan x, \quad [0, \frac{\pi}{4}], \quad \text{and} \quad N = 4$
15. $f(x) = e^{x^3}, \quad [0, 1], \quad \text{and} \quad N = 4$
16. $f(x) = \frac{2e^{-x^2}}{\sqrt{\pi}}, \quad [0, 3], \quad \text{and} \quad N = 10$
17. $f(x) = \frac{e^{-x}}{x}, \quad [0.5, 3], \quad \text{and} \quad N = 8$
18. $f(x) = e^{-\sin x}, \quad [0, \frac{\pi}{2}], \quad \text{and} \quad N = 6$
19. $f(x) = \frac{2^x}{x+1}, \quad [0, 1], \quad \text{and} \quad N = 4$
20. $f(x) = \sqrt{1 + e^x}, \quad [0, 1], \quad \text{and} \quad N = 4$
21. $f(x) = \sqrt{\sin x}, \quad [0, \frac{\pi}{2}], \quad \text{and} \quad N = 4$
22. $f(x) = \sqrt{1 - 2\sin^2 x}, \quad [0, \frac{\pi}{2}], \quad \text{and} \quad N = 4$

In the following problems find N so that the midpoint rule approximates the given $\int$ with $|E_N| <$ given ε. If $N \leq 10$ use the midpoint rule to find the approximate value.

23. $\int_{1.5}^{3} \frac{dx}{x}, \quad \varepsilon = \pm 10^{-3}$
24. $\int_{0}^{\sqrt{\pi}} \sin x^2 dx, \quad \varepsilon = \pm 10^{-3}$
25. $\int_{0}^{1} \frac{dx}{1 + x^4}, \quad \varepsilon = \pm 10^{-4}$
26. $\int_{\frac{\pi}{2}}^{\pi} \sqrt{1 + \cos^2 x}\, dx, \quad \varepsilon = \pm 10^{-3}$
27. $\int_{1}^{4} e^{\sqrt{x}} dx, \quad \varepsilon = \pm 10^{-2}$
28. $\int_{9}^{13} \arctan \sqrt{x}\, dx, \quad \varepsilon = \pm 10^{-4}$
29. $\int_{0}^{\sqrt{\frac{\pi}{2}}} \cos(x^2) dx, \quad \varepsilon = \pm 10^{-2}$
30. $\int_{0}^{\frac{\pi}{2}} \sqrt{1 + \cos^2 x}\, dx, \quad \varepsilon = \pm 10^{-2}$

31. $\displaystyle\int_0^1 \sqrt{1+x^3}\,dx \qquad \varepsilon = \pm 10^{-2}$

32. $\displaystyle\int_0^{\frac{1}{2}} \frac{dx}{1+x^2}, \qquad \varepsilon = \pm 10^{-4}$

33. $\displaystyle\int_0^{\frac{1}{2}} \frac{dx}{\sqrt{1-x^2}}, \qquad \varepsilon = \pm 10^{-3}$

34. $\displaystyle\int_1^3 4^{-x}dx, \qquad \varepsilon = \pm 10^{-2}$

35. $\displaystyle\int_1^2 \frac{\sin x}{x}\,dx, \qquad \varepsilon = \pm 10^{-2}$

13.2 Trapezoidal Rule

This is another way to approximate $\int_a^b f(x)dx$. Again divide $[a,b]$ into N equal intervals $\big($of length $h = \frac{b-a}{N}\big)$.

The trapezoid approximation to the integral is given by

$$T_N = \frac{h}{2}\left\{f(a) + 2f(x_1) + 2f(x_2) + \ldots + 2f(x_{N-1}) + f(b)\right\}.$$

We write

$$\int_a^b f(x)dx = \underbrace{\qquad T_N \qquad}_{\text{Trap. approximation}} + \underbrace{\quad E_N \quad}_{\text{error}}.$$

If B is chosen so that $|f''(z)| \le B$ for all $z \in [a,b]$ then

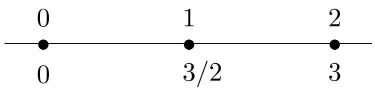
$$|E_N| \le \frac{(b-a)^3 B}{12N^2}.$$

Note: This is not the same as for the midpoint rule.

13.3 Algorithm for Computing T_{2N} from T_N

If we have computed T_N we can get T_{2N} as in the following example for $\int_0^3 f(x)dx$:

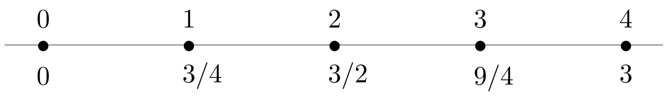
$N = 2:$



$(h = 3/2)$

$$T_2 = \frac{3}{2}\left\{\frac{1}{2}f(0) + f\left(\frac{3}{2}\right) + \frac{1}{2}f(3)\right\}.$$

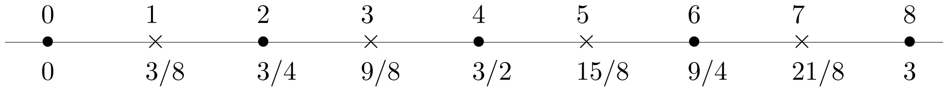
$N = 4:$



$(h_4 = 3/4)$

$$\begin{aligned}
T_4 &= \frac{3}{4}\left\{\frac{1}{2}f(0)+f\left(\frac{3}{4}\right)+f\left(\frac{3}{2}\right)+f\left(\frac{9}{4}\right)+\frac{1}{2}f(3)\right\}\\
T_4 &= \frac{1}{4}\left\{3\left(\frac{1}{2}f(0)+f\left(\frac{3}{2}\right)+\frac{1}{2}f(3)\right)+3\left(f\left(\frac{3}{4}\right)+f\left(\frac{9}{4}\right)\right)\right\}\\
T_4 &= \frac{1}{4}\left\{2T_2+3\left(f\left(\frac{3}{4}\right)+f\left(\frac{9}{4}\right)\right)\right\}. \qquad \text{(Look back at the formula for } T_2\text{)}\\
T_4 &= \frac{1}{2}T_2+h_4\left(f\left(\frac{3}{4}\right)+f\left(\frac{9}{4}\right)\right)
\end{aligned}$$

$N = 8:$



$\left(h_8 = \frac{3}{8}\right)$

$$\begin{aligned}
T_8 &= \frac{3}{8}\left\{\frac{1}{2}f(0)+f\left(\frac{3}{8}\right)+f\left(\frac{3}{4}\right)+f\left(\frac{9}{8}\right)++f\left(\frac{3}{2}\right)+f\left(\frac{15}{8}\right)+f\left(\frac{9}{4}\right)+f\left(\frac{21}{8}\right)+\frac{1}{2}f(3)\right\}\\
T_8 &= \frac{1}{8}\left\{3\left(\frac{1}{2}f(0)+f\left(\frac{3}{4}\right)+f\left(\frac{3}{2}\right)+f\left(\frac{9}{4}\right)+\frac{1}{2}f(3)\right)+\right.\\
&\qquad \left.3\left(f\left(\frac{3}{8}\right)+f\left(\frac{9}{8}\right)+f\left(\frac{15}{8}\right)+f\left(\frac{21}{8}\right)\right)\right\}\\
T_8 &= \frac{1}{8}\left\{4T_4+3\left(f\left(\frac{3}{8}\right)+f\left(\frac{9}{8}\right)+f\left(\frac{15}{8}\right)+f\left(\frac{21}{8}\right)\right)\right\}\\
T_8 &= \frac{1}{2}T_4+h_8\left(f\left(\frac{3}{8}\right)+f\left(\frac{9}{8}\right)+f\left(\frac{15}{8}\right)+f\left(\frac{21}{8}\right)\right)
\end{aligned}$$

13.4 Algorithm

I. Partition interval of integration into N intervals and compute the points $x_0, x_2, \ldots, x_N$ of the partition.

II. Compute T_N and record.

III. Compute midpoints, y_i, of the partition to give a new partition with $2N$ intervals. (of length $h_{2N} = (b-a)/2N$).

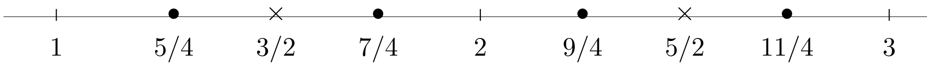
IV. Compute $T_{2N} = \frac{1}{2}T_N + \frac{b-a}{2N}\sum_{i=0}^{N-1} f(y_i)$, record and store.

V. Reset N to $2N$ and repeat steps III to V.

<u>Example</u>:

For $\int_1^3 \sin\left(\frac{1}{x}\right)\,dx$, compute T_2, T_4, and T_8.

Solution: (using a T.I. calculator)



Since $h_2 = \frac{3-1}{2} = 1$,

$$T_2 = \frac{1}{2}\sin(1)\ \boxed{+}\ \sin\left(\frac{1}{2}\right)\ \boxed{+}\ \frac{1}{2}\sin\left(\frac{1}{3}\right)\ \boxed{=}\ (1.0637584).$$

Since $h_4 = \frac{3-1}{4} = \frac{1}{2}$,

$$\begin{aligned} T_4 &= \boxed{\div}\ 2 \quad \left(\text{Now } \tfrac{1}{2}T_2 \text{ is in the display.}\right) \\ &\quad \boxed{+}\ \boxed{(}\ \sin\left(\frac{2}{3}\right)\ \boxed{+}\ \sin\left(\frac{2}{5}\right)\ \boxed{)}\ \boxed{\div}\ 2\ \boxed{=}\ (1.035773). \end{aligned}$$

Since $h_8 = \frac{3-1}{8} = \frac{1}{4}$,

$$\begin{aligned} T_8 &= \boxed{\div}\ 2 \quad \left(\text{Now } \tfrac{1}{2}T_4 \text{ is in the display}\right)\ \boxed{+}\ \boxed{(}\ \sin\left(\frac{4}{5}\right) \\ &\quad \boxed{+}\ \sin\left(\frac{4}{7}\right)\ \boxed{+}\ \sin\left(\frac{4}{9}\right)\ \boxed{+}\ \sin\left(\frac{4}{11}\right)\ \boxed{)}\ \boxed{\div}\ 4\ \boxed{=}\ (1.0288421). \end{aligned}$$

Problems

36. For $\displaystyle\int_1^3 \sin\left(\frac{1}{x}\right) dx$ compute T_{16}.

37. For $\displaystyle\int_1^3 e^x dx$ compute T_2, T_4, T_8, and T_{16}.

38. For $\displaystyle\int_0^{\frac{1}{2}} \cos x\, dx$ compute T_2, T_4, T_8, and T_{16}.

39. For $\displaystyle\int_0^1 e^{-1/x} dx$ compute T_2, T_4, T_8, and T_{16}.

40. For $\displaystyle\int_1^2 \frac{dx}{x}$ compute T_2, T_4, T_8, and T_{16}.

41. For $\displaystyle\int_0^1 e^{-x^2} dx$ compute T_2, T_4, and T_8.

42. For $\displaystyle\int_0^1 (x^2+1)^{\frac{3}{2}} dx$ compute T_2, T_4, and T_8.

43. For $\displaystyle\int_0^7 (1+x)^{\frac{1}{3}} dx$ compute T_2, T_4, and T_8.

44. For $\displaystyle\int_0^{\frac{\pi}{2}} \sqrt{1 - 2\sin^2 x}\ dx$ compute T_2, T_4, and T_8.

45. For $\int_0^{\frac{\pi}{2}} \sqrt{\sin x}\ dx$ compute T_2, T_4, and T_8.

46. For $\int_0^1 \frac{2^x}{x+1}\ dx$ compute T_2, T_4, and T_8.

For the integral $\int_a^b f(x)dx$ find a suitable N and compute T_N (if $N \leq 10$) such that $|E_N| \leq$ (given ε).

47. $\int_1^2 \sin\left(\frac{1}{x}\right) dx,\quad \varepsilon \leq .002$

48. $\int_1^2 \frac{dx}{x},\quad \varepsilon \leq .007$

49. $\int_0^1 4x^3 dx,\quad \varepsilon \leq .06$

50. $\int_0^{\frac{\pi}{2}} \sqrt{1+\cos^2 x}\ dx,\quad \varepsilon \leq .02$

51. $\int_0^1 \sqrt{1+x^3}\ dx,\quad \varepsilon \leq .02$

52. $\int_9^{13} \arctan\sqrt{x}\, dx,\quad \varepsilon \leq .001$

53. $\int_1^3 e^{\sqrt{x}} dx,\quad \varepsilon \leq .05$

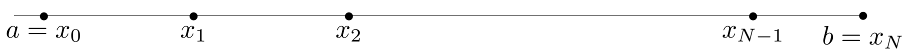
54. $\int_0^{\sqrt{\pi/2}} \cos(x^2) dx,\quad \varepsilon \leq .01$

55. $\int_1^2 \frac{\sin x}{x}\ dx,\quad \varepsilon \leq .01$

56. $\int_1^3 e^{-x} dx,\quad \varepsilon \leq .01$

13.5 Simpson's Rule

There is another way to approximate $\int_a^b f(x)\ dx$. For this rule partition $[a,b]$ into an *even number* N of equal subintervals, each of length $h = \frac{b-a}{N}$ to get a partition



Then the Simpson's approximation is given by

$$S_N = \frac{b-a}{3N}\left[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \cdots + 2f(x_{N-2}) + 4f(x_{N-1}) + f(x_N)\right].$$

Simpson's Rule can be remembered as

$$\frac{b-a}{3N}\left[(\text{ends}) + 4(\text{odds}) + 2(\text{evens})\right].$$

Thus

$$\int_a^b f(x)dx = S_N + E_N\ .$$

If B is chosen so that $|f^{(4)}(z)| \leq B$ for all $z \in [a,b]$ then

$$|E_N| \leq \frac{(b-a)^5 B}{180N^4}.$$

Problems

Use Simpson's rule to estimate, with $|E_N| \leq$ (given ε) the following integrals.

57. $\displaystyle\int_1^{1.9} \frac{dx}{x}$, $\varepsilon = .001$

58. $\displaystyle\int_1^{3} 5^{-x} dx$, $\varepsilon = .001$

59. $\displaystyle\int_1^{2} \frac{\sin x}{x}\, dx$, $\varepsilon = .001$

60. $\displaystyle\int_1^{3} e^{\sqrt{x}} dx$, $\varepsilon = .001$

61. $\displaystyle\int_0^{\frac{1}{2}} \cos x\, dx$, $\varepsilon = 1 \times 10^{-6}$

62. $\displaystyle\int_0^{\frac{1}{2}} \frac{dx}{\sqrt{1-x^2}}$, $\varepsilon = 1 \times 10^{-6}$

63. $\displaystyle\int_{e^2}^{10} \frac{dx}{x}$, $\varepsilon = 5 \times 10^{-6}$

64. $\displaystyle\int_0^{\frac{1}{2}} \frac{dx}{1+x^2}$, $\varepsilon = 1 \times 10^{-7}$

65. $\displaystyle\int_0^{1} \left(1 + \frac{1}{1+x^2}\right)^{\frac{1}{2}} dx$, $\varepsilon = .05$

66. $\displaystyle\int_1^{2} \frac{dx}{1+x}$, $\varepsilon = 2 \times 10^{-6}$

67. $\displaystyle\int_0^{\frac{\pi}{2}} \cos x^2\, dx$, $\varepsilon = .001$

13.6 A Little Numerical Analysis

Sometimes we cannot use the error formula to estimate the error. In these circumstances we compute the sequence

$$T_2, T_4, T_8, \ldots, T_N, T_{2N}, \ldots$$

of trapezoidal approximations, and stop when the difference $D_N = T_{2N} - T_N$ gets small. We do it this way for two reasons:

1. There is a good algorithm to compute T_{2N} using T_N.
2. If we know T_N and T_{2N} we can get the Simpson's approximation by the formula

$$S_N = \frac{4T_{2N} - T_N}{3} = T_{2N} + \frac{T_{2N} - T_N}{3} = T_{2N} + \frac{D_N}{3}.$$

13.7 Algorithm for Computing S_N and D_N

I. Partition the interval of integration $[a, b]$ into N subintervals as described on page 191.

II. Calculate T_N; store and print.

III. Compute midpoints, y_i, of your partition.

IV. Compute $\frac{b-a}{2N}\{f(y_0) + f(y_1) + \cdots + f(y_{N-1})\} = A$.

V. Compute $T_{2N} = \frac{1}{2}T_N + A$; store and print.

VI. Compute $D_N = T_{2N} - T_N$ and print.

VII. Compute $S_N = T_{2N} + \frac{D_N}{3}$.

PROBLEMS

68. For $\int_0^{\frac{\pi}{4}} \cos x \, dx$, compute T_{16}, D_8, and S_8 and compare T_{16} and S_8 with the exact value of this integral.

In problems 69 to 82 do the following.

For $N = 2, 4$ and the given integral calculate T_N, T_{2N}, D_N, and S_N. (Include the partition, your method, and a summary of the results in your solution.)

69. $\int_1^5 \frac{dx}{x}$

70. $\int_0^1 e^x \, dx$

71. $\int_0^{\frac{\pi}{4}} \sec x \, dx$

72. $\int_0^1 \frac{dx}{1+x^2}$

73. $\int_0^{\frac{\pi}{2}} \cos x^2 \, dx$

74. $\int_0^1 e^{\sqrt{x}} \, dx$

75. $\int_0^{\frac{\pi}{2}} \sqrt{\sin x} \, dx$

76. $\int_0^1 \frac{dx}{1+x^3}$

77. $\int_1^9 \sqrt{3+x^3} \, dx$

78. $\int_1^9 \sqrt{x^3-1} \, dx$

79. $\int_0^4 \frac{2^x}{x+1} \, dx$

80. $\int_2^5 \frac{dx}{\ln x}$

81. $\int_2^4 \frac{x \, dx}{\ln x}$

82. $\int_0^{\frac{\pi}{4}} x \tan x \, dx$