Contents

Chapter 14

Taylor's Theorem

14.1 Polynomials

Every polynomial can be expressed not only in powers of x but also in powers of $x - a$.

Example: Write $x^2 + x + 2$ in powers of $x - 1$.

Method 1:

Set $w = x - 1$. Then $x = w + 1$ and
$x^2 + x + 2 = (w + 1)^2 + (w + 1) + 2 = w^2 + 3w + 4.$
Therefore $x^2 + x + 2 = (x - 1)^2 + 3(x - 1) + 4.$
This method is not practical if the degree of the polynomial is greater than 3.

Method 2:

Write $2 + x + x^2 = C_0 + C_1(x - 1) + C_2(x - 1)^2.$ $(*)$

Determine the coefficients as follows:

1. Evaluate $(*)$ at $x = 1$ to obtain $C_0 = 4$.
2. Differentiate both sides of $(*)$:
   $1 + 2x = C_1 + 2C_2(x - 1).$ $(**)$
3. Evaluate $(**)$ at $x = 1$ to get $C_1 = 3$.
4. Differentiate both sides of $(**)$:
   $2 = 2C_2 \Rightarrow C_2 = 1.$
   Therefore $2 + x + x^2 = 4 + 3(x - 1) + (x - 1)^2.$

The general case:

Let $p(x)$ be a polynomial of degree n and write

$$p(x) = C_0 + C_1(x - a) + C_2(x - a)^2 + C_3(x - a)^3 + \cdots + C_n(x - a)^n \quad (1)$$

The algorithm for computing the coefficients C_i is:

(0) $p(a) \quad = C_0$

(1) $p'(x) \quad = C_1 + 2C_2(x-a) + 3C_3(x-a)^2 + \cdots + nC_n(x-a)^{n-1}$

$p'(a) \quad = C_1$

(2) $p''(x) \quad = 2C_2 + 2\cdot 3C_3(x-a) + \cdots + n(n-1)C_n(x-a)^{n-2}$

$p''(a) \quad = 2C_2 \Rightarrow C_2 = p''(a)/2$

(3) $p'''(x) = 2\cdot 3C_3 + 2\cdot 3\cdot 4C_4(x-a) + \cdots + (n-2)(n-1)nC_n(x-a)^{n-2}$

$p'''(a) = 2\cdot 3C_3 \Rightarrow C_3 = p'''(a)/2\cdot 3$

$\vdots$

(n) $p^{(n)}(x) = n!C_n$ and $p^{(n)}(a) = n!C_n$

Therefore $C_n = p^{(n)}(a)/n!$

Now substitute the values of C_i into (1) to obtain the formula:

$$\begin{aligned} p(x) \quad = \quad & p(a) + p'(a)(x-a) + \frac{p''(a)}{2}(x-a)^2 + \frac{p'''(a)}{3!}(x-a)^3 \\ & + \frac{P^{(4)}(a)}{4!}(x-a)^4 + \cdots + \frac{p^n(a)}{n!}(x-a)^n \end{aligned}$$

or

$$p(x) = \sum_{k=0}^{n} \frac{p^{(k)}(a)}{k!}(x-a)^k \qquad \text{Note: } p^{(0)} = p.$$

Problems

1. Express $p(x) = x^3 - x^2 + 1$ in powers of $x - \frac{1}{2}$ by method 1 and method 2.
2. Express $p(x) = x^3 - x - 2$ in powers of $x + 1$ by method 1 and method 2.
3. Express $p(x) = x^4 - x - 1$ in powers of $x + 2$ by method 1 and method 2.
4. Express $p(x) = x^5 + x^3 + x^2 - 1$ in powers of $x - 5$.

Expand the following polynomials in powers of $x - a$:

5. $(x+1)^2 - x + 2$;	$a = 1$	6. $x^3 - 3x^4 + 4x$;	$a = -2$
7. $2x^3 + 5x^2 + 13x + 10$;	$a = -1$	8. $x^4 - 5x^2 + x + 2$;	$a = 2$
9. $2x^4 + 5x^3 + 4x + 16$;	$a = -2$	10. $3x^3 - 2x^2 - 2x + 1$;	$a = 1$
11. $5x^5 + 4x^4 - 3x^3 - 2x^2 + x + 1$;	$a = -1$	12. $x^5 + 2x^4 + 3x^2 + 4x + 5$;	$a = -2$
13. $x^4 - 7x^3 + 5x^2 + 3x - 6$;	$a = 0$	14. $(x^2-1)^4 + 3(x^2-1)^2 + x^2 - 1$;	$a = 1$

15. (a) $(x-1)^4$; $a=0$ (b) $(x-1)^6$; $a=0$
    (c) $(x+1)^5$; $a=0$ (d) $(x-1)^5+3(x-1)^2-x+2$; $a=0$

Application: Evaluating polynomials.

Example: Let $p(x)=x^3-x^2+1$. Compute $p(0.50028)$ to 5 places.

Solution:

Use method 2 to write

$$\begin{aligned} p(x) &= \frac{7}{8}-\frac{1}{4}\left(x-\frac{1}{2}\right)+\frac{1}{2}\left(x-\frac{1}{2}\right)^2+\left(x-\frac{1}{2}\right)^3 \\ p(0.50028) &= p\left(\frac{1}{2}+0.00028\right) \\ &= \frac{7}{8}-\frac{1}{4}(0.00028)+\frac{1}{2}(0.00028)^2+(0.00028)^3. \end{aligned}$$

The last two terms are less than 10^{-7}.

Therefore to 5 places $p(0.50028)$ is

$$\begin{aligned} \frac{7}{8}-\frac{1}{4}(0.00028) &= 0.87500-0.00007 \\ &= 0.87493. \end{aligned}$$

Expand the following polynomials in powers of $x-a$ for an appropriate value of a and evaluate to 4 significant digits for the given value of x.

PROBLEMS

16. x^3-3x^2+2x+1; $x=1.004$
17. $x^5+x^4+x^3+x^2+x+1$; $x=1.994$
18. $4x^4-3x^2+10x+12$; $x=-0.9890$
19. $10x^3+12x^2-6x-5$; $x=-3.042$

14.1.1 Polynomials with Specific Properties

Write down a polynomial $p(x)$ in $(x-a)$, which has the properties listed. Try to make the number of terms as small as possible.

Example: $a=-1$; $p(-1)=0$; $p^{(3)}(-1)=36$.

Solution: Since $p(x)$ can be written in the form:

$$\begin{aligned} p(x) &= p(-1)+p'(-1)(x+1)+\frac{p''(-1)}{2}(x+1)^2+\frac{p^{(3)}(-1)}{3!}(x+1)^3 \\ &\quad +\frac{p^{(4)}(-1)}{4!}(x+1)^4+\cdots \end{aligned}$$

We see that $p(x) = 0 + \frac{36}{6}(x+1)^3$ and $p(x) = (x+1) + 6(x+1)^3 - \pi(x+1)^5$ are two polynomials with the required properties. We choose the polynomial $6(x+1)^3$ because it has the fewest terms. To avoid mistakes compute $p(-1)$ and $p^{(3)}(-1)$ for your choice of $p(x)$.

Problems

20. (a) $a = 1$; $f(1) = 5$; $f^{(4)}(1) = 168$
    (b) $a = 0$; $f(1) = 5$; $f^{(4)}(1) = 168$
    (c) $a = 0$; $f(1) = 5$; $f^{(3)}(1) = f^{(4)}(1) = 168$
21. (a) $a = -1$; $f(-1) = 1$; $f^{(3)}(-1) = 0$; $f^{(4)}(-1) = 48$
    (b) $a = 0$; $f(0) = 1$; $f'(0) = 0$; $f''(0) = -1$; $f^{(3)}(0) = 0$
    (c) $a = 1$; $f(1) = 0$; $f'(1) = 1$; $f''(1) = -1$; $f^{(3)}(1) = 2$
22. (a) $a = 0$; $f(0) = 0$; $f'(0) = 1$; $f''(0) = 0$; $f^{(3)}(0) = 1$; $f^{(4)}(0) = 0$ $f^{(5)}(0) = 1$
    (b) $a = 2$; $f'(2) = 1$; $f^{(3)}(2) = -1$; $f^{(5)}(2) = 1$
    (c) $a = \pi$; $f(\pi) = 1$; $f^{(2)}(\pi) = 1$; $f^{(4)}(\pi) = 4!$

14.2 Successive Differentiation

In the following problems a function f, and a number c are given.

1. Compute the first 5 derivatives of f and evaluate at $x = c$. *Organize your work as shown in the example below.*
2. Look for a common pattern for these derivatives. In order to see if there is any pattern in the calculation do *not* multiply the factors together.
3. If you can find a common pattern write down a formula for $f^{(n)}(x)$ and $f^{(n)}(c)$.

Example: $f(x) = \frac{4}{x} - \frac{1}{x^2}$; $k = 5$; $c = 1$

1. *Computations:*

$f(x) = \frac{4}{x} - \frac{1}{x^2}$	$f(1) = 4 - 1 = 3$
$f'(x) = -\frac{4}{x^2} + \frac{2}{x^3}$	$f'(1) = -4 + 2 = -2$
$f''(x) = \frac{4\cdot 2}{x^3} - \frac{2\cdot 3}{x^4}$	$f''(1) = 8 - 6 = 2$
$f^{(3)}(x) = -\frac{4\cdot 2\cdot 3}{x^4} + \frac{2\cdot 3\cdot 4}{x^5}$	$f^{(3)}(1) = -24 + 24 = 0$
$f^{(4)}(x) = \frac{4\cdot 2\cdot 3\cdot 4}{x^5} - \frac{2\cdot 3\cdot 4\cdot 5}{x^6}$	$f^{(4)}(1) = 96 - 120 = -24$
$f^{(5)}(x) = -\frac{4\cdot 2\cdot 3\cdot 4\cdot 5}{x^6} + \frac{2\cdot 3\cdot 4\cdot 5\cdot 6}{x^7}$	$f^{(5)}(1) = -480 + 720 = 240$

2. *Pattern:*

$$f(x) = 4\left(\frac{1}{x}\right) - \frac{1}{x^2}$$

$$f'(x) = -4\left(\frac{1}{x^2}\right) + \frac{2!}{x^3}$$

$$f''(x) = 4\left(\frac{2!}{x^3}\right) - \frac{3!}{x^4}$$

$$f^{(3)}(x) = -4\left(\frac{3!}{x^4}\right) + \frac{4!}{x^5}$$

$$f^{(4)}(x) = 4\left(\frac{4!}{x^5}\right) - \frac{5!}{x^6}$$

$$f^{(5)}(x) = -4\left(\frac{5!}{x^6}\right) + \frac{6!}{x^7}$$

3. *Formula:*

$$f^{(n)}(x) = 4(-1)^n \frac{n!}{x^{n+1}} - \frac{(-1)^n(n+1)!}{x^{n+2}}$$

$$f^{(n)}(x) = (-1)^n \frac{n!}{x^{n+2}}(4x - n - 1)$$

$$f^n(1) = (-1)^n n!(3 - n)$$

Problems

23. (a) $f(x) = (x-3)^5;\ c = 6$ (b) $f(x) = (x+1)^n;\ c = 2$

24. (a) $f(x) = \frac{1}{x};\ c = 3$ (b) $f(x) = \frac{1}{x^p};\ c = 2$

25. (a) $f(x) = \frac{1}{x-a};\ c = 2a$ (b) $f(x) = \frac{-1}{x+a};\ c = 3a$

26. (a) $f(x) = x^{\frac{1}{3}};\ c = 2$ (b) $f(x) = x^{\frac{1}{5}};\ c = 32$

27. (a) $f(x) = (x+1)^{\frac{1}{2}};\ c = 1$ (b) $f(x) = (1-x)^{\frac{1}{4}};\ c = \frac{1}{2}$

28. (a) $f(x) = \frac{1}{1-x} + \frac{1}{1+x};\ c = 3$ (b) $f(x) = \frac{x}{1-x^2};\ c = \frac{1}{4}$

29. (a) $f(x) = \frac{x}{x+1};\ c = 1$ (b) $f(x) = \frac{1}{(1-x)^2};\ c = \frac{1}{4}$

30. (a) $f(x) = \frac{1}{(1-x)^5};\ c = \frac{1}{4}$ (b) $f(x) = \frac{1}{(1+x)^4};\ c = 3$

31. (a) $f(x) = \frac{x}{(1-x)^2};\ c = \frac{1}{2}$ (b) $f(x) = \frac{1}{1+x^2};\ c = 2$

32. (a) $f(x) = 2^x;\ c = 6$ (b) $f(x) = e^{\frac{x}{2}};\ c = 4$

33. (a) $f(x) = e^{ax};\ c = \frac{3}{a}$ (b) $f(x) = xe^{-x};\ c = \frac{1}{2}$

34. (a) $f(x) = e^{-x}\sin x;\ c = \frac{\pi}{2}$ (b) $f(x) = e^{-x}\cos x;\ c = \pi$

35. (a) $f(x) = \ln x;\ c = 10$ (b) $f(x) = \ln ax;\ c = 3$

36. (a) $f(x) = x\ln x;\ c = 2$ (b) $f(x) = \ln\left(\frac{x-1}{x+1}\right);\ c = \frac{1}{4}$

37. (a) $f(x) = \frac{1}{2}(e^x - e^{-x});\ c = 6$ (b) $f(x) = \frac{1}{2}(e^x + e^{-x});\ c = 5$

38. (a) $f(x) = \sin 2x;\ c = \frac{\pi}{4}$ (b) $f(x) = 3\cos x;\ c = \frac{1}{2}$

39. (a) $f(x) = \sin(x-1);\ c = 3$ (b) $f(x) = \cos(2x+1);\ c = \pi$

40. (a) $f(x) = \sin x - \cos x;\ c = \frac{1}{2}$ (b) $f(x) = \sin x \cos x;\ c = 2$

41. (a) $f(x) = \sin^2 x;\ c = \frac{\pi}{2}$ (b) $f(x) = 2\cos^2 x - 1;\ c = \pi$

14.3 Taylor Polynomials

Definition:

The nth Taylor polynomials of $f(x)$ at $x = a$ is

$$f_{n,a}(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k; \quad f^{(0)}(a) = f(a)$$

and $f^{(k)}(a)$ is the kth derivative of f at $x = a$.

Observations:

1. The function must have n derivatives in order for the function to have a Taylor polynomial of degree n.
2. $f_{n,a}^{(k)}(a) = f^{(k)}(a)$ for $k = 1, 2, \ldots, n$.
3. If $f^{(n)}(a) = 0$, then of course $f_{n,a}(x)$ exists but the degree of the polynomial for $f_{n,a}(x)$ is at most $n-1$. Thus the degree of the nth Taylor polynomial can be $< n$. (It is always $\leq n$.)

Compute $f_{n,a}(x)$ for the given f, n, and a.

42. $f(x) = \frac{1}{1-x};\ n = n;\ a = 0$

43. $f(x) = \frac{1-x^{p+1}}{1-x};\ n = p;\ a = 0$

44. $f(x) = \frac{1}{1+x^2};\ n = 4;\ a = 0$

45. (a) $f(x) = \sin 2x;\ n = 7;\ a = 0$
    (b) $f(x) = \sin 2x;\ n = 6;\ a = 0$

46. (a) $f(x) = \sin 2x;\ n = 7;\ a = \frac{\pi}{4}$
    (b) $f(x) = \cos 2x;\ n = 7;\ a = \frac{\pi}{4}$

47. $f(x) = xe^x;\ n = 4;\ a = 1$

48. $f(x) = x\sin x;\ n = 4;\ a = \frac{\pi}{2}$

49. $f(x) = x^2 e^{-x};\ n = 5;\ a = 1$

50. $f(x) = \sin x + \cos x;\ n = 5;\ a = 0$

51. $f(x) = 1 + e^x + e^{2x};\ n = 10;\ a = 0$

52. $f(x) = x^2 \ln x;\ n = 3;\ a = e$

53. $f(x) = \displaystyle\int_0^x \frac{dt}{1+t};\ n = 5;\ a = 0$

54. $f(x) = \displaystyle\int_0^x \frac{dt}{1+t^2};\ n = 4;\ a = \frac{\pi}{4}$

55. $f(x) = \displaystyle\int_0^x e^{-t^2} dt;\ n = 5;\ a = 0$

56. $f(x) = \displaystyle\int_0^x \frac{dt}{\sqrt{1-t^2}};\ n = 4;\ a = 1$

57. $f(x) = \displaystyle\int_0^x \sin^4 t\, dt;\ n = 5;\ a = \frac{\pi}{4}$

58. $f(x) = \displaystyle\int_1^x 2^t dt;\ n = 5;\ a = 1$

59. $f(x) = \ln\left(\frac{1+x}{1-x}\right);\ n = 5;\ a = 0$

60. $f(x) = \sqrt{1+x}\,;\ n;\ a = 0$

61. $f(x) = 10^x;\ n;\ a = 0$

62. $f(x) = \tan x;\ n = 2;\ a = 0$

63. $f(x) = \arcsin x;\ n = 2;\ a = 0$

64. $f(x) = \arccos x;\ n = 2;\ a = 0$

65. $f(x) = \arctan x;\ n = 2;\ a = 0$

66. $f(x) = \sin x - x\cos x;\ n = 2;\ a = 0$

67. $f(x) = (1+x)^\alpha;\ n;\ a = 0$

14.4 Properties of Taylor Polynomials

Let f and g be functions; c_1 and c_2 are constants

1. $(c_1 f + c_2 g)_{n,a} = c_1 f_{n,a} + c_2 g_{n,a}$.
2. $(f_{n,a})' = (f')_{n-1,a}$.
3. Set $g(x) = \displaystyle\int_a^x f(t)dt$. Then $(g(x))_{n+1,a} = \displaystyle\int_a^x f_{n,a}(t)dt$.
4. Set $g(x) = f(cx)$. Then $g_{n,a}(x) = f_{n,ca}(cx)$.
5. Let p_n be a polynomial of degree $n \geq 1$.

Let f and g be two functions with derivatives of order n at 0.

Let $f(x) = p_n(x) + x^n g(x)$ where $\lim\limits_{x\to 0} g(x) = 0$.

Then p_n is the Taylor polynomial of f at 0.

Example 1: For $f(x) = \frac{1}{1-x}$ compute $f_{5,0}(x)$.

Solution A: Use the definition of the 5th degree Taylor polynomial; i.e.,

$$f_{5,0}(x) = \sum_{k=0}^{5} \frac{f^{(k)}(0)}{k!}\, x^k.$$

First computer $f(0) = 1$, $f'(0) = 1$, $f''(0) = 2!$, $f^{(3)}(0) = 3!$, $f^{(4)}(0) = 4!$, and $f^{(5)}(0) = 5!$. Therefore $f_{5,0}(x) = 1 + x + x^2 + x^3 + x^4 + x^5$.

Solution B:

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + x^5 + \frac{x^6}{1-x} \qquad \text{(by long division)}$$

and so $f(x) = 1 + x + x^2 + x^4 + x^5 + x^5 g(x)$ where $g(x) = \frac{x}{1-x}$.

Since $\lim\limits_{x\to 0} g(x) = 0$, we may use property 5 to conclude that $f_{5,0}(x) = 1 + x + x^2 + x^4 + x^5$.

Example 2: For $h(x) = \frac{1}{1+x}$ compute $h_5(x)$ at $x = 0$.

Solution: $h(x) = \frac{1}{1-(-x)}$.

So $h(x) = f(-x)$ where $f(x) = \frac{1}{1-x}$; that is, $h(x) = f(cx)$ with $c = -1$. Thus we may use property 4: $h_{(5,0)} = f_{5,0}(-x) = 1 - x + x^2 - x^3 + x^4 - x^5$ by substitution of $-x$ into $f_{5,0}$ done in Example 1.

Example 3: Let $\psi(x) = \frac{x^2}{1-x^2}$, compute $\psi_{5,0}(x)$.

Solution:

$$\frac{x^2}{1-x^2} = -1 + \frac{1}{1-x^2} = -1 + \frac{1/2}{1-x} + \frac{1/2}{1+x}.$$

Property 1 allows us to use the results of Examples 1 and 2 to write

$$\begin{aligned}
\psi(x) &= -1 + \frac{1}{2}(1 + x + x^2 + x^3 + x^4 + x^5 + 1 - x^2 + x^2 - x^3 + x^4 - x^5) \\
\psi(x) &= 0 + 0x + x^2 + 0x^3 + x^4 + 0x^5 \\
\psi(x) &= x^2 + x^4.
\end{aligned}$$

Note: It would be quite messy to do this problem by computing derivatives.

Example 4: Let $g(x) = \frac{1}{(1+x)^2}$, compute $g_{4,0}(x)$.

Solution: Observe that $g(x) = -\left(\frac{1}{1+x}\right)' = -h'(x)$.

Thus

$$g_{4,0}(x) = -\big(h'(x)\big)_{4,0} \qquad \text{(property 1)}$$

and

$$\big(h'(x)\big)_{4,0} = \big(h_{5,0}(x)\big)'. \qquad \text{(property 2)}$$

But

$$h_{5,0}(x) = 1 - x + x^2 - x^3 + x^4 - x^5. \qquad \text{(from Example 2)}$$

Thus

$$\begin{aligned}
\big(h_{5,0}(x)\big)' &= -1 + 2x - 3x^2 + 4x^3 - 5x^4 \\
g_{4,0}(x) &= -(-1 + 2x - 3x^2 + 4x^3 - 5x^4) \\
g_{4,0}(x) &= 1 - 2x + 3x^2 - 4x + 5x^4.
\end{aligned}$$

Example 5: Let $f(x) = \ln(x+1)$ compute $f_{5,0}(x)$.

Solution: Observe that $\ln(x+1) = \displaystyle\int_0^x \frac{dt}{1+r}$.

Set $h(t) = \frac{1}{1+t}$ and so $h_{4,0}(t) = 1 - t + t^2 - t^3 + t^4$ by Example 2 again. Since $f_{5,0}(x) = \displaystyle\int_0^x h_{4,0}(t)dt$ by property 3 we conclude that

$$\begin{aligned}
f_{5,0}(x) &= t - \frac{t^2}{2} + \frac{t^3}{3} - \frac{t^4}{4} + \frac{t^5}{5}\Bigg|_0^x \\
f_{5,0}(x) &= x - \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \frac{x^5}{5}.
\end{aligned}$$

14.5 Taylor's Theorem

Introduction

The polynomial function, $p(x) = a_0 + a_1 x + \cdots + a_n x^n$ can be computed easily for any number x. Recall that, Horner's method makes this easy calculation efficient. This is not true for functions f like $\sin, \log, a^x$, or $f(x) = \displaystyle\int_0^x e^{-t^2}\,dt$.

However the Taylor polynomial $f_{n,a}$ of f has the property that

$$f^{(k)}(a) = f_{n,a}^{(k)}(a); \quad k = 1, 2, \ldots, n.$$

This means, as we shall see, that near $x = a$ the Taylor polynomial is an excellent approximation for $f(x)$.

14.5.1 Error Term $E_{n,a}(x)$

We write

$$f(x) = f_{n,a}(x) + E_{n,a}(x) \qquad \text{(approximation + error)}$$

Thus

$$E_{n,a}(x) = f(x) - f_{n,a}(x).$$

To see how good an approximation $f_{n,a}(x)$ is we need an estimate for the error term $E_{n,a}(x)$. This is provided by

Taylor's Theorem:

Assume $f(x)$ has $(n+1)$ derivatives in an interval I containing a. Suppose M_{n+1} is a constant chosen so that for every $z \in I$:

$$\left|f^{(n+1)}(z)\right| \le M_{n+1}.$$

Then

$$\left|E_{n,a}(x)\right| \le \frac{M_{n+1}}{(n+1)!}|x-a|^{n+1}.$$

PROBLEMS

Given $f(x), a, n,$ and I compute $f_{n,a}(x)$ and find a value for M_{n+1}. (You are, of course, required to give reasons to support your choice for M_{n+1}.)

Example: $f(x) = \frac{1}{1-x}$, $a = -1$, $n = 5$ and $I = [-2, 0]$.

Solution:

$f(x) = \frac{1}{1-x}$ $\qquad f(-1) = \frac{1}{2}$

$f'(x) = \frac{1}{(1-x)^2}$ $\qquad f'(-1) = \frac{1}{4}$

$f''(x) = \frac{2}{(1-x)^3}$ $\qquad f''(-1) = \frac{1}{4}$

$f'''(x) = \frac{2\cdot 3}{(1-x)^4}$ $\qquad f'''(-1) = \frac{3}{8}$

$f^{(4)}(x) = \frac{2\cdot3\cdot4}{(1-x)^5}$ $\qquad f^{(4)}(-1) = \frac{3}{4}$

$f^{(5)}(x) = \frac{5!}{(1-x)^6}$ $\qquad f^{(5)}(-1) = \frac{15}{8}$

Since

$$\begin{aligned} f_{5,-1}(x) &= f(-1) + f'(-1)(x+1) + \frac{f''(-1)}{2!}(x+1)^2 + \\ &\quad \frac{f'''(-1)}{3!}(x+1)^3 + \frac{f^{(4)}(-1)}{4!}(x+1)^4 + \frac{f^{(5)}(-1)}{5!}(x+1)^5, \\ f_{5,-1}(x) &= \frac{1}{2} + \frac{x+1}{4} + \frac{(x+1)^2}{4\cdot 2!} + \frac{3(x+1)^3}{8\cdot 3!} + \frac{3(x+1)^4}{4\cdot 4!} + \frac{15(x+1)^5}{8\cdot 5!}. \end{aligned}$$

Next, write

$$g(z) = \left|f^{(6)}(z)\right| = \frac{6!}{(1-z)^7}; \quad z \in [-2,0].$$

Since $g'(z) = \frac{7!}{(1-x)^8} > 0$ on $[-2,0]$, $g(z)$ is an increasing function on $[-2,0]$. Because $g(0) = 6!$ is the maximum value of g on $[-2,0]$ we choose $M_6 = 6!$ (Note reasons for your choice of M_{n+1} must be given in your solution.)

68. $f(x) = \frac{1}{1+x^2}$; $a = 1,\ n = 2$, and $I = [0,2]$

69. $f(x) = \sin 2x$; $a = \frac{\pi}{4},\ n = 5$, and $I = \left[0, \frac{\pi}{2}\right]$

70. $f(x) = \frac{1}{x+1} - \frac{2}{x-1}$; $a = 3,\ n = 4$, and $I[2,4]$

71. $f(x) = \frac{1}{x^2-1}$; $a = 0,\ n = 5$ and $I = \left[-\frac{1}{2}, \frac{1}{2}\right]$

Hint: Expand $f(x)$ by partial fractions and then differentiate.

72. $f(x) = \sqrt{1+x}$; $a = 0,\ n = 5$, and $I = \left[-\frac{1}{2}, \frac{1}{2}\right]$

73. $f(x) = xe^x$; $a = 1,\ n = 5$, and $I = \left[\frac{1}{2}, \frac{3}{2}\right]$

74. $f(x) = \ln(x+1)$; $a = 0,\ n = 5$, and $I = \left[-\frac{1}{2}, \frac{1}{2}\right]$

75. $f(x) = x\ln x$; $a = 3,\ n = 6$, and $I = [2,4]$

76. $f(x) = \ln\left(\frac{x-1}{x+1}\right)$; $a = 3,\ n = 5$, and $I = \left[2\frac{1}{2}, 3\frac{1}{2}\right]$

77. $f(x) = \frac{1}{2}(e^x - e^{-x})$; $a = 1,\ n = 5$, and $I = [0,2]$

Given $f(x)$, a, ε and I find n such that $\left|E_{n,a}(x)\right| < \varepsilon$.

<u>Example</u>: $f(x) = \frac{1}{1-x}$, $a = -1$, $\varepsilon = .001$, and $I = \left[-\frac{3}{2}, -\frac{1}{2}\right]$.

Solution:

From our work on the previous example w see that

$$f^{(n+1)}(x) = \frac{(n+1)!}{(1-x)^{n+2}}.$$

This function is increasing in I (check the derivative !) and positive. Thus

$$\begin{aligned} \left|f^{(n+1)}(x)\right| &= f^{(n+1)}(x)\frac{(n+1)!}{(1-1-\frac{1}{2})^{n+2}} \\ &= \frac{(n+1)!}{\left(\frac{3}{2}\right)^{n+2}} = \left(\frac{2}{3}\right)^{n+2}(n+1)! \end{aligned}$$

Set $M_{n+1} = \left(\frac{2}{3}\right)^{n+2}(n+1)!$

Thus

$$\left|E_{n,-1}(x)\right| \le \frac{|x+1|^{n+1}}{\left(\frac{3}{2}\right)^{n+2}} = \frac{2}{3}\left|\frac{2}{3}(x+1)\right|^{n+1}.$$

Since $\frac{2}{3}(x+1)$ is an increasing function on I the maximum is $\frac{1}{3}$ and the minimum is $-\frac{1}{3}$ on I. Therefore $\left|\frac{2}{3}(x+1)\right| \le \frac{1}{3}$ on I. Thus

$$\left|E_{n,-1}(x)\right| \le \frac{2}{3}\left(\frac{1}{3^{n+1}}\right) = \frac{2}{3^{n+2}}.$$

We need only choose n so that $\frac{2}{3}\left(\frac{1}{3^{n+1}}\right) < .001$ in order to guarantee that $\left|E_{n,-1}(x)\right| < .001$.

Now, for $\frac{2}{3^{n+2}} < .001$ try $n = 5$:

$\frac{2}{3^7} \cong .00091$ and for $n = 4$ we have $\frac{2}{3^6} \cong .0027$. *Therefore $n = 5$ works but $n = 4$ does not.*

78. $f(x) = (x+1)e^x$; $a = 1$, $\varepsilon = 5 \times 10^{-4}$, $I = \left[\frac{1}{2}, \frac{3}{2}\right]$

79. $f(x) = \sin x$; $a = \frac{\pi}{2}$, $\varepsilon = 5 \times 10^{-3}$, $I = \left[\frac{\pi}{4}, \frac{3\pi}{4}\right]$

80. $f(x) = \sin x + cosx$; $a = \pi$, $\varepsilon = 5 \times 10^{-3}$, $I = \left[\frac{3\pi}{4}, \frac{5\pi}{4}\right]$

81. $f(x) = x\ln x$; $a = 3$, $\varepsilon = 5 \times 10^{-3}$, $I = \left[\frac{5}{2}, \frac{7}{2}\right]$

82. $f(x) = \frac{1}{2}(e^x + e^{-x})$; $a = 0$, $\varepsilon = 5 \times 10^{-3}$, $I = \left[-\frac{1}{2}, \frac{1}{2}\right]$

83. $f(x) = \sin x\, \cos x$; $a = \frac{\pi}{4}$, $\varepsilon = 1 \times 10^{-2}$, $I = \left[\frac{\pi}{8}, \frac{3\pi}{8}\right]$

84. $f(x) = \frac{1}{x+1} + \frac{1}{x-1}$; $a = 2$, $\varepsilon = 1 \times 10^{-3}$, $I = \left[\frac{3}{2}, \frac{5}{2}\right]$

85. $f(x) = \sqrt{x} + 2x^2 - 1$; $a = \frac{1}{2}$, $\varepsilon = 1 \times 10^{-2}$, $I = \left[\frac{1}{4}, \frac{3}{4}\right]$

86. $f(x) = \ln(x+1)$; $a = 1$, $\varepsilon = 1 \times 10^{-3}$, $I = \left[\frac{3}{4}, \frac{5}{4}\right]$

14.6 Approximations by Taylor Polynomials

Definition

An approximation is said to be accurate to k decimal places if $|E| < 5 \times 10^{-(k+1)}$ where E is the error.

If we wish to approximate a function f at some fixed point x by a Taylor polynomial we must choose a near to x such that $f^{(k)}(a)$ is easy to compute.

<u>Example 1</u>: For $f(x) = e^{-(2x+1)}$ approximate $f(-2)$, $f(0)$, and $f(-1)$ to two decimal places.

Solution:

Since $f^{(n)}(x) = (-1)^n 2^n / e^{2x+1}$ and $e^{2x+1} \neq 0$ for all x, $f^{(n)}(a)$ exists for any choice of a, n, and interval containing a. Pick $a = -\frac{1}{2}$. This is the only choice for a if we wish to compute $f^{(n)}(a)$ exactly. Thus $f^{(n)}\left(-\frac{1}{2}\right) = (-1)^n 2^n$.

14.6.1 Error Estimate for $f(-2)$

Choose $I = \left[-2, -\frac{1}{2}\right]$. For $z \in I$,

$$\left|f^{(n+1)}(z)\right| = \frac{2^{n+1}}{e^{2z+1}}$$

because $e^{2z+1} > 0$ for all $z \in \mathbb{R}$. e^{2z+1} is an increasing function for all $z \in \mathbb{R}$. Therefore

$$\left|f^{(n+1)}(z)\right| \leq \frac{2^{n+1}}{e^{-3}}.$$

(We minimized the denominator on I to get a bound on $\left|f^{(n+1)}(z)\right|$.

This is not a very useful bound since the function we wish to approximate appears in our formula for $\left|f^{(n+1)}(z)\right|$. But $2 < e < 3$ implies that $2^{n+1}e^3 < 2^{n+1}3^3$. Pick $M_{n+1} = 2^{n+1} \cdot 27$ to give

$$\begin{aligned} \left|E_{n,-\frac{1}{2}}(x)\right| &\leq \frac{2^{n+1} \cdot 27|x + \frac{1}{2}|^{n+1}}{(n+1)!} \\ &\leq \frac{27|2x+1|^{n+1}}{(n+1)!}. \end{aligned}$$

Thus

$$\left|E_{n,-\frac{1}{2}}(-2)\right| \leq \frac{27 \cdot 3^{n+1}}{(n+1)!}.$$

Therefore we must solve $\frac{27 \cdot 3^{n+1}}{(n+1)!} < 5 \times 10^{-3}$ for n to guarantee the actual error is less than 2 decimal places.

Calculations

$$\begin{aligned} n = 5 &: \quad \frac{3^6}{6!} = \frac{729}{720} \cong 1.0125 = s_1 \\ n = 6 &: \quad \frac{3^7}{7!} \cong s_1 \times \frac{3}{7} \cong .4339 = s_2 \\ n = 7 &: \quad \frac{3^8}{8!} \cong s_2 \times \frac{3}{8} \cong .1627 = s_3 \\ n = 8 &: \quad \frac{3^9}{9!} \cong s_3 \times \frac{3}{9} \cong .0542 = s_4 \\ n = 9 &: \quad \frac{3^{10}}{10!} \cong s_4 \times \frac{3}{10} \cong .0163 = s_5 \\ n = 10 &: \quad \frac{3^{11}}{11!} \cong s_5 \times \frac{3}{11} \cong .0044 = s_6 \end{aligned}$$

Now $.0044 < 5 \times 10^{-3}$, so $f_{10,-\frac{1}{2}}(-2)$ approximates $f(2)$ to two decimal places.

14.6.2 Error Estimate for $f(0)$

Choose $I = \left[-\frac{1}{2}, 0\right]$. The analysis of our first case applies here so we choose the endpoint, $-\frac{1}{2}$, to maximize $\left|f^{(n+1)}(z)\right|$ on I; this is,

$$\left|f^{(n+1)}(z)\right| \leq \left|f^{(n+1)}(-\frac{1}{2})\right| = 2^{n+1}.$$

Therefore

$$\begin{aligned} \left|E_{n,-\frac{1}{2}}(x)\right| &\leq \frac{2^{n+1}|x+\frac{1}{2}|^{n+1}}{(n+1)!} \\ &\leq \frac{|2x+1|^{n+1}}{(n+1)!} \end{aligned}$$

Thus $\left|E_{n,-\frac{1}{2}}(0)\right| \leq \frac{1}{(n+1)!}$. Set $\frac{1}{(n+1)!} < 5 \times 10^{-3}$ and solve for n. For $n = 4$, $\frac{1}{5!} < .008$ and $n = 5$ gives $\frac{1}{6!} < .0014$. Therefore $n = 5$ works and $f_{5,-\frac{1}{2}}(0)$ approximates $f(0)$ to two decimal places.

$$\begin{aligned} f_{5,-\frac{1}{2}}(0) &= f\left(-\frac{1}{2}\right) + f'\left(-\frac{1}{2}\right)\left(\frac{1}{2}\right) + \frac{f''\left(-\frac{1}{2}\right)}{2}\left(\frac{1}{2}\right)^2 + \frac{f'''\left(-\frac{1}{2}\right)}{3!}\left(\frac{1}{2}\right)^3 \\ &\quad + \frac{f^{(4)}\left(-\frac{1}{2}\right)}{4!}\left(\frac{1}{2}\right)^4 + \frac{f^{(5)}\left(-\frac{1}{2}\right)}{5!}\left(\frac{1}{2}\right)^5 \\ &= 1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} \cong .36666667. \end{aligned}$$

Therefore .37 approximates $f(0)$ to 2 decimal places.

14.6.3 Error Estimate for $f(-1)$

Choose $I = \left[-1, -\frac{1}{2}\right]$. The analysis of our first case applies for all intervals. Therefore we choose the endpoint, -1, to maximize $\left|f^{(n+1)}(z)\right|$ on I; that is,

$$\left|f^{(n+1)}(z)\right| \leq \left|f^{(n+1)}(-1)\right| = 2^{n+1}e < 2^{n+1} \cdot 3.$$

Therefore

$$\left|E_{n,-\frac{1}{2}}(x)\right| \leq \frac{2^{n+1} \cdot 3|x+\frac{1}{2}|^{n+1}}{(n+1)!} \leq \frac{3|2x+1|^{n+1}}{(n+1)!}.$$

Thus $\left|E_{n,-\frac{1}{2}}(-1)\right| \leq \frac{3}{(n+1)!}$. Set $\frac{3}{(n+1)!} < 5 \times 10^{-3}$. For $n = 5$, $\frac{3}{6!} < .0042$. Therefore $n = 5$ works and $f_{5,-\frac{1}{2}}(-1)$ approximates $f(-1)$ to two decimal places

$$\begin{aligned} f_{5,-\frac{1}{2}}(-1) &= f\left(-\frac{1}{2}\right) + f'\left(-\frac{1}{2}\right)\left(-1+\frac{1}{2}\right) + \frac{f''\left(-\frac{1}{2}\right)}{2!}\left(-1+\frac{1}{2}\right)^2 + \cdots + \frac{f^{(5)}\left(-\frac{1}{2}\right)}{5!}\left(-1+\frac{1}{2}\right) \\ &= 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} \\ &\cong 2.7166667. \end{aligned}$$

Therefore 2.72 approximates $f(-1)$ to two decimal places.

Example 2: Determine $\sqrt[3]{e}$ with an error of less than $\pm 10^{-5}$.

Solution:

Because $f\left(\frac{1}{3}\right) = \sqrt[3]{e}$ for the function $f(x) = e^x$ and $f^{(n)}(x) = e^x$ can be evaluated exactly at $x = 0$ we approximate $\sqrt[3]{e}$ by computing $f_{n,0}(x)$ at $x = 3$ for a suitable value of n. That is,

$$f(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + E_{n,0}(x)$$

and

$$f(x) \cong f_{n,0}(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!}.$$

Next we compute

$$\left|E_{n,0}(x)\right| \leq \frac{M_{n+1}}{(n+1)!}|x|^{n+1}.$$

We find M_{n+1}, a bound for $\left|f^{(n+1)}(z)\right|$ on $\left[0, \frac{1}{3}\right]$, as follows:

$\left|f^{(n+1)}(z)\right| = e^z$ and since e^z is increasing on $\left[0, \frac{1}{3}\right]$

$\left|f^{(n+1)}(z)\right| \leq e^{\frac{1}{3}}$. But $e^{\frac{1}{3}} < 2$ since $e < 3$.

This fact allows us to choose $M_{n+1} = 2$. Therefore

$$\left|E_{n,0}\left(\frac{1}{3}\right)\right| \leq \frac{2}{(n+1)!3^{n+1}}.$$

For $n = 4$, $\left|E_{4,0}\left(\frac{1}{3}\right)\right| \leq \frac{2}{5!3^5} < .000069$.

For $n = 5$, $\left|E_{5,0}\left(\frac{1}{3}\right)\right| \leq \frac{2}{6!3^6} < .000004 < 10^{-5}$.

Hence $\sqrt[3]{e}$ is approximated by $f_{5,0}\left(\frac{1}{3}\right)$ with an error of $\pm 10^{-5}$.

$$\sqrt[3]{e} \cong 1 + \frac{1}{3} + \frac{1}{2!3^2} + \frac{1}{3!3^3} + \frac{1}{4!3^4} + \frac{1}{5!3^5} \cong 1.3956104.$$

87. For $f(x) = \sin x$ what values of n are required to compute $f(.2)$, $f(.6)$, and $f(1)$ with an accuracy within 10^{-5} using the Taylor polynomial $f_{n,0}(x)$?

88. For $f(x) = \ln(x+1)$ what values of n are required to compute $f(.1)$, $f(.4)$, and $f(.8)$ with an accuracy within 10^{-5} using the Taylor polynomial $f_{n,0}(x)$?

89. For $f(x) = \sqrt{x}$ what values of n are required to compute $f(15)$, $f(17)$ and $f(18)$ with an accuracy within 10^{-4} using the Taylor polynomial $f_{n,16}(x)$?

In the following problems compute the given quantity to the given accuracy ε using an appropriate Taylor polynomial.

90. $\ln 2$, $\varepsilon = \pm 1 \times 10^{-4}$

91. $\ln \frac{3}{2}$, $\varepsilon = \pm 1 \times 10^{-3}$

92. $\ln(1.25)$, $\varepsilon = \pm 1 \times 10^{-3}$

93. $\sin .7$, $\varepsilon = \pm 1 \times 10^{-4}$

94. $\cos .8$, $\varepsilon = \pm 1 \times 10^{-4}$

95. $\cos 1.5$, $\varepsilon = \pm 1 \times 10^{-4}$

96. $(8.5)^{\frac{1}{3}}$, $\varepsilon = \pm 1 \times 10^{-3}$

97. $(82)^{\frac{1}{3}}$, $\varepsilon = \pm 1 \times 10^{-3}$

98. $(30)^{\frac{1}{5}}$, $\varepsilon = \pm 1 \times 10^{-4}$

99. $(0.91)^{\frac{1}{3}}$, $\varepsilon = \pm 1 \times 10^{-4}$

100. $(15)^{\frac{1}{4}}$, $\varepsilon = \pm 1 \times 10^{-4}$

101. $(1.08)^{\frac{1}{4}}$, $\varepsilon = \pm 1 \times 10^{-4}$

102. $\sqrt{10}$, $\varepsilon = \pm 1 \times 10^{-4}$

103. $e^{-0.2}$, $\varepsilon = \pm 1 \times 10^{-3}$

104. $e^{.2}$, $\varepsilon = \pm 1 \times 10^{-4}$

105. $\ln(0.8)$, $\varepsilon = \pm 1 \times 10^{-4}$

106. $7^{\frac{1}{3}}$, $\varepsilon = \pm 1 \times 10^{-4}$

107. $(1.1)^{\frac{1}{5}}$, $\varepsilon = \pm 1 \times 10^{-4}$

14.7 Series

Comparison Test

Suppose Σa_n and Σb_n are positive term series for $n > N$; i.e., only a finite number of terms of the two series are negative. Then

(i) if $a_n \leq b_n$ for $n > N$ and Σb_n converges we can conclude that Σa_n converges also.

(ii) if $a_n \geq b_n$ for $n > N$ and Σb_n diverges we can conclude that Σa_n diverges also.

Remarks:

(i) If $a_n \leq b_n$ we say the series Σb_n dominates Σa_n or Σa_n is dominated by Σb_n. In these terms the comparison test can be described as follows:

If a given positive series is dominated by a convergent series, then it converges; if it dominates a divergent series, then it diverges.

(ii) In order to apply this test we need a to have a collection of series whose convergence or divergence is known. You can start your own collection with the two following series.

1. The p-series $\sum\limits_{n=1}^{\infty} \frac{1}{n^p}$

   converges if $p > 1$

   diverges if $p \leq 1$.

2. The geometric series $\sum\limits_{n=0}^{\infty} ar^n$, $a \neq 0$ converges if $|r| < 1$ to the sum $\frac{a}{1-r}$ diverges if $|r| \geq 1$.

Examples

(1) $\sum\limits_{n=1}^{\infty} \frac{1}{2+5^n}$.

In order to apply the comparison test we need some insight in how this series is related to a known convergent or divergent series. Now $2+5^n$ is approximately equal to 5^n for large values of n. Therefore when n is large $\frac{1}{2+5^n}$ behaves like $\frac{1}{5^n}$ and $\frac{1}{5^n} = \left(\frac{1}{5}\right)^n$. This is the nth term of the convergent geometric series with $a = 1$ and $r = \frac{1}{5}$. Now we can apply the comparison test with $a_n = \frac{1}{2+5^n}$ and $b_n = \frac{1}{5^n}$.

Since $2 + 5^n > 5^n$, $\frac{1}{2+5^n} < \frac{1}{5^n}$. Therefore $\Sigma\frac{1}{5^n}$ dominates $\Sigma\frac{1}{2+5^n}$ and $\Sigma\frac{1}{5^n}$ is a convergence geometric series so $\Sigma\frac{1}{2+5^n}$ converges by the comparison test.

(2) $\sum_{n=1}^{\infty} \frac{1}{2n-1} = 1 + \frac{1}{3} + \frac{1}{5} + \cdots$.

For large values of n it is clear that $\frac{1}{2n-1}$ behaves like $\frac{1}{2n}$. Now the series $\Sigma\frac{1}{2n} = \frac{1}{2}\Sigma\frac{1}{n}$ diverges since $\Sigma\frac{1}{n}$ is a p series for $p = 1$.

Since $2n - 1 < 2n$, $\frac{1}{2n-1} > \frac{1}{2n}$. Thus $\sum_{n=1}^{\infty} \frac{1}{2n-1}$ dominates a divergent series and by the comparison test must diverge also.

(3) $\sum_{n=1}^{\infty} \frac{1}{n!} = 1 + \frac{1}{2} + \frac{1}{2\cdot 3} + \frac{1}{2\cdot 3\cdot 4} + \frac{1}{2\cdot 3\cdot 4\cdot 5} + \cdots$.

For $n \geq 2$, $n! = 2\cdot 3\cdot 4\cdots n \geq \underbrace{2\cdots 2}_{\substack{n-1 \\ \text{terms}}} = 2^{n-1}$. Now $n! \geq 2^{n-1}$ implies $\frac{1}{n!} < \frac{1}{2^n-1}$. So $\sum_{n=1}^{\infty} \frac{1}{n!}$ is dominated by $\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^{n-1}$ a convergent geometric series. Hence $\sum_{n=1}^{\infty} \frac{1}{n!}$ converges by the comparison test.

(4) $\sum_{K=1}^{\infty} \frac{K^2+1}{K^3+1}$.

For large values of K, $\frac{K^2+1}{K^3+1}$ looks like $\frac{K^2}{K^3}$ or $\frac{1}{K}$. Therefore we should try to show that $\frac{K^2+1}{K^3+1} > \frac{1}{K}$ to conclude $\sum_{K=1}^{\infty} \frac{K^2+1}{K^3+1}$ diverges by the comparison test.

Now $\frac{K^2+1}{K^3+1} > \frac{K^2}{K^3+1} > \frac{K^2}{K^3+K^3} = \frac{1}{2K}$. The first inequality holds because $K^2 + 1 > K^2$ the second is true because $K^3 + 1 < K^3 + K^3$. Since $\Sigma\frac{1}{2K} = \frac{1}{2}\Sigma\frac{1}{K}$ is a divergent series. $\Sigma\frac{K^2+1}{K^3+1}$ diverges by the comparison test as predicted.

Note: The required inequality can be *proved* in more than one way. For example:

Since $K^3 + 1 \leq K^3 + K$ we can write

$$\frac{K^2+1}{K^3+1} > \frac{K^2+1}{K^3+K} = \frac{K^2+1}{K(K^2+1)} = \frac{1}{K} \quad (K > 1).$$

Problems

Use the comparison test to establish the convergence or divergence of $\sum_{n=1}^{\infty} f(n)$ for the given $f(n)$.

108. $f(n) = \frac{1}{1+\ln(n)}$

109. $f(n) = \frac{1}{n^2-1} \quad (n \neq 1) \quad f(1) = 0$

110. $f(n) = \frac{\sqrt{n}}{n+4}$

111. $f(n) = \frac{2^n}{n+1}$

112. $f(n) = \frac{n-1}{n^2+1}$

113. $f(n) = \frac{1}{n\sqrt{n}}$

114. $f(n) = \frac{n^2}{n^4+3n+1}$

115. $f(n) = \frac{n+1}{n^3}$

116. $f(n) = \frac{8}{3^n+n^2}$

117. $f(n) = \frac{n+2}{n^2+1}$

118. $f(n) = \frac{n+3}{n!}$

119. $f(n) = \frac{1}{\sqrt{n}} - \left(\frac{1}{n}\right)$

120. $f(n) = \frac{\ln(n)}{n^3}$

121. $f(n) = \frac{1}{2n+1}\left(\frac{1}{4}\right)^n$

122. $f(n) = \frac{1+e^{-n}}{e^n}$

123. $f(n) = \sin\left(\frac{1}{n^2}\right)$

124. $f(n) = \frac{1}{n5^n}$

125. $f(n) = \frac{n+1}{(n+2)3^n}$

126. $f(n) = \frac{1}{n-3} - \frac{1}{n}$

127. $f(n) = \frac{\sin^2 n}{2^n}$

128. $f(n) = \frac{1}{\sqrt{n^2+3n}}$

129. $f(n) = \frac{2+\cos n}{7^n}$

130. $f(n) = \left(30 + \frac{2}{n}\right)\frac{1}{2^n}$

131. $f(n) = \frac{2^n+3}{4^n+5}$

132. $f(n) = \frac{3+\cos n}{n}$

133. $f(n) = \frac{1}{n^n}$

134. $f(n) = \frac{1}{n3^{n-1}}$

135. $f(n) = \frac{1}{\sqrt{n^2-4}}$

14.7.1 Ratio Test

Let Σa_n be an infinite series of non zero terms.

(i) If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L < 1$, the series is absolutely convergent.

(ii) If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L > 1$, or $L = \infty$ the series is divergent.

(iii) If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1$, the test gives *no information* about the given series.

Examples

(1) $\sum_{n=0}^{\infty}(2^n+5)/3^n$.

Since $\left|\frac{a_{n+1}}{a_n}\right| = \frac{a_{n+1}}{a_n}$ we compute

$$\frac{(2^{n+1}+5)/3^{n+1}}{(2^n+5)/3^n} = \frac{1}{3}\left(\frac{2^{n+1}+5}{2^n+5}\right) = \frac{1}{3}\left(\frac{2+5/2^n}{1+5/2^n}\right).$$

This last step is the result of multiplying the numerator and denominator of $\frac{2^{n+1}+5}{2^n+5}$ by $\frac{1}{2^n}$. Next observe that $\lim_{n\to\infty}\frac{1}{3}\left(\frac{2+5/2^n}{1+5/2^n}\right) = \frac{2}{3}$. Therefore the series converges since $L = \frac{2}{3} < 1$. In this example we can actually compute the sum of the series as follows: (Remember the ratio test can only detect convergence or divergence providing $L \neq 1$ not what the series converges to.)

$$\sum_{r=0}^{\infty}\frac{2^n+5}{3^n} = \sum_{n=0}^{\infty}\left(\frac{2}{3}\right)^n + 5\sum_{n=0}^{\infty}\frac{1}{3^n} = \frac{1}{1-\frac{2}{3}} = \frac{5}{1-\frac{1}{3}} = \frac{21}{2}.$$

(2) $2 + 2^2/2^8 + 2^3/3^8 + 2^4/4^8 + \cdots$.

(a) Find $a_n : a_n = 2^n/n^8$.

(b) Compute $\left|\frac{a_{n+1}}{a_n}\right| : \frac{2^{n+1}/(n+1)^8}{2^n/n^8} = \frac{2^{n+1}}{(n+1)^8}\cdot\frac{n^8}{2^n} = 2\left(\frac{n}{n+1}\right)^8$.

(c) Compute $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| : \lim_{n\to\infty} 2\left(\frac{n}{n+1}\right)^8 = 2\left(\lim_{n\to\infty}\frac{n}{n+1}\right)^8 = 2$.

(d) The series diverges since $L = 2 > 1$.

Problems

Use the ratio test to conclude what you can about the convergence or divergence of $\sum_{n=2}^{\infty} f(n)$ for each given $f(n)$.

136. $f(n) = \frac{3^n}{n^3}$

137. $f(n) = \frac{(-1)^n 3^{n-2}}{2^n}$

138. $f(n) = \frac{n}{(n+1)e^n}$

139. $f(n) = \frac{n7^n}{n!}$

140. $f(n) = \frac{10^{2n}}{(2n-1)!}$

141. $f(n) = \frac{(-1)^n 3^{n-1}}{n!}$

142. $f(n) = \frac{2^{3n}}{3^{2n}}$

143. $f(n) = n\left(\frac{2}{3}\right)^n$

144. $f(n) = \frac{\left(\sqrt{5}-1\right)^n}{n^2+1}$

145. $f(n) = \frac{n^2+7n+1}{n^2 n!}$

146. $f(n) = \frac{n^n}{3^n n!}$

147. $f(n) = \frac{n^n}{2^n n!}$

148. $f(n) = \frac{5^n n!}{n^n}$

149. $f(n) = \frac{5^n n!}{(2n)^n}$

150. $f(n) = \frac{n!}{1 \cdot 3 \cdot 5 \ldots (2n-1)}$

151. $f(n) = \frac{n(n+2)}{(n+1)!}$

152. $f(n) = \frac{e^n}{n!}$

153. $f(n) = \frac{n!}{100^n}$

154. $f(n) = \frac{(2n)!}{n!n!}$

155. $f(n) = \frac{3^n}{(n!)^a}\,, \quad a > 0$

156. $f(n) = \frac{1 \cdot 3 \cdot 5 \cdots (2K-1)}{1 \cdot 4 \cdot 7 \cdots (3K-2)}$

14.7.2 Root Test

Let Σa_n be an infinite series.

(i) If $\lim\limits_{n\to\infty} \left(|a_n|^{\frac{1}{n}}\right) = L < 1$, the series is absolutely convergent.

(ii) If $\lim\limits_{n\to\infty} \left(|a_n|^{\frac{1}{n}}\right) = L > 1$ or ∞, the series is divergent.

(iii) If $\lim\limits_{n\to\infty} \left(|a_n|^{\frac{1}{n}}\right) = 1$, the test gives no information on the behaviour of the series.

Problems

Use the root test to test for convergence or divergence of the following series.

157. $\sum_{n=1}^{\infty} \frac{1}{n^n}$

158. $\sum_{n=1}^{\infty} \frac{2^n}{n^2}$

159. $\sum_{n=1}^{\infty} \left(1 + \frac{1}{n}\right)^n$

160. $\sum_{n=1}^{\infty} \frac{(-1)^n}{[\ln(n+1)]^n}$

161. $\sum_{n=1}^{\infty} (-1)^{n+1} \left(\frac{n}{3n+1}\right)^n$

162. $\sum_{n=1}^{\infty} \frac{n^n}{\left(2n+\frac{1}{n}\right)^n}$

163. $\sum_{n=2}^{\infty} \frac{(-1)^n n^n}{(\ln n)^n}$

164. $\sum_{n=1}^{\infty} \left(\frac{n}{n+1}\right)^{n^2}$

14.7.3 The Integral Test

If $a_n = f(n)$ for $n \geq J$ and f is a positive nonincreasing function for $x \geq J$ (J a positive integer) then the infinite series $\sum\limits_{n=1}^{\infty} a_n$

(a) converges if $\displaystyle\int_J^{\infty} f(x)dx$ converges

(b) diverges if $\displaystyle\int_J^{\infty} f(x)dx$ diverges.

Examples

1. $\sum\limits_{n=2}^{\infty} \frac{1}{n \ln n}$.

   Set $f(x) = \frac{1}{x \ln x}$ since $f(n) = \frac{1}{n \ln n} = a_4$ for $n > 1$. For $x > 1$, $f(x) > 0$ and $f'(x) = \frac{-(1+\ln x)}{x^2 \ln^2 x}$ is *negative*. Therefore f is positive, decreasing and continuous for $x > 1$ meeting the conditions required for the integral test.

   $\displaystyle\int_2^{\infty} \frac{1}{x \ln x} dx = \lim_{n\to\infty} \int_2^{n} \frac{1}{n \ln x} dx = \lim_{n\to\infty} (\ln(\ln x)) \Big|_2^n$.

   $\displaystyle\int_2^{\infty} \frac{1}{x \ln x} dx = \lim_{n\to\infty} \{\ln(\ln n) - \ln(\ln 2)\} = \infty$. Therefore the series diverges.

2. $\sum\limits_{n=1}^{\infty} \frac{1}{n^2}$.

   Set $f(x) = \frac{1}{x^2}$. $f(x) > 0$ and since $f'(x) > 0$ and since $f'(x) = -\frac{2}{x^3} f(x)$ is decreasing for all $x \geq 1$. Also $f(n) = \frac{1}{n^2}$. Therefore we can apply the integral test using $f(x)$:

   $$\int_1^{\infty} \frac{dx}{x^2} = \lim_{n\to\infty} \int_1^{n} \frac{dx}{x^2} = \lim_{n\to\infty} \left(-\frac{1}{x}\right) \Big|_1^n = \lim_{n\to\infty} \left(-\frac{1}{n} + 1\right) = 1$$

   to conclude that $\sum\limits_{n=1}^{\infty} \frac{1}{n^2}$ converges.

Problems

Use the integral test to determine whether the following series converge or diverge.

165. $\sum\limits_{n=1}^{\infty} \frac{1}{(3+2n)^2}$

166. $\sum\limits_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$

167. $\sum\limits_{n=1}^{\infty} \frac{1}{4n+5}$

168. $\sum\limits_{n=1}^{\infty} \frac{100}{(4+n)^{\frac{3}{2}}}$

169. $\sum\limits_{n=1}^{\infty} \frac{\ln n}{n}$

170. $\sum\limits_{n=1}^{\infty} \frac{1}{1+3n^2}$

171. $\sum\limits_{n=1}^{\infty} \frac{\ln n}{n^2}$

172. $\sum\limits_{n=2}^{\infty} \frac{1}{(\ln n)^3}$

173. $\sum_{n=1}^{\infty} \frac{n}{e^n}$

174. $\sum_{n=1}^{\infty} n3^{-n^2}$

175. $\sum_{n=1}^{\infty} \frac{\arctan n}{1+n^2}$

176. $\sum_{n=4}^{\infty} \frac{n}{\ln n}$

177. $\sum_{n=1}^{\infty} \frac{5^{\frac{1}{n}}}{n^2}$

178. $\sum_{n=1}^{\infty} \frac{1}{n^2} \sin \frac{\pi}{n}$

179. $\sum_{n=1}^{\infty} \frac{e^{\arctan n}}{1+n^2}$

180. $\sum_{n=2}^{\infty} \frac{1}{n \ln n \ln(\ln n)}$

181. $\sum_{n=1}^{\infty} \ln\left(1+\frac{1}{n}\right)$

182. $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^{\frac{1}{4}}}$

183. $\sum_{n=1}^{\infty} \frac{n}{2^n}$

184. $\sum_{n=1}^{\infty} \frac{1}{n(n+1)(n+2)}$

14.7.4 Alternating Series Test

The series $\sum_{n=1}^{\infty} (-1)^{n+1} a_n = a_1 - a_2 + a_3 - a_4 + \cdots$ converges if

(a) $a_n > 0$ for all n

(b) $a_n \geq a_{n+1}$ for every n

(c) $\lim_{n\to\infty} a_n = 0$.

14.7.5 Alternating Series Approximation Theorem

If $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$ $(a_n > 0)$ converges with sum S and s_n is the nth partial sum then

$$|S - s_n| < a_{n+1} \qquad \text{for all } n.$$

Examples:

1. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots$

   Since:

   (a) $a_n = \left|\frac{(-1)^{n+1}}{n}\right| > 0$,

   (b) $|a_{n+1}| = \frac{1}{n+1} < \frac{1}{n}$, and

   (c) $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n} = 0$.

   the series converges.

2. $1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{8} + \frac{1}{4} + \frac{1}{16} + \cdots$.

   In this example the alternating series test fails because $a_{n+1} < a_n$ is *not* true for all n.

PROBLEMS

Determine whether the following series are convergent.

185. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{(n+3)}{n(n+2)}$.

Hint: to establish part (b) define a function $f(x)=\frac{x+3}{x(x+2)}$ and *show* this function is decreasing.

186. $\sum_{n=1}^{\infty}\frac{(-1)^n}{n!}$

187. $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}n}{n^3+2}$

188. $\sum_{n=1}^{\infty}\frac{-\cos n}{n^3}$

189. $\sum_{n=3}^{\infty}\frac{(-1)^{n+1}\sqrt{n}}{n+3}$

190. $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{\sqrt{2n+1}}$

191. $\sum_{n=2}^{\infty}(-1)^n\frac{\ln n}{n}$

192. $\sum_{n=2}^{\infty}(-1)^n\frac{n}{\ln n}$

193. $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n^{\frac{2}{3}}}$

194. $\sum_{n=1}^{\infty}(-1)^{n+1}\sin\frac{\pi}{n}$

195. $\sum_{n=1}^{\infty}\ln n\cos n\pi$

196. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\ln(n+1)}{n\sqrt{n}}$

197. $\sum_{n=0}^{\infty}(-1)^n\frac{1+4^n}{1+3^n}$

198. $\sum_{n=0}^{\infty}(-1)^n\frac{\sqrt{n+1}}{8n+5}$

199. $\sum_{n=1}^{\infty}(-1)^n\frac{e^n}{n^4}$

200. $\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}$

201. $\sum_{n=45}^{\infty}(-1)^{n+1}\frac{1}{\ln(\ln n)}$

202. $\sum_{n=0}^{\infty}(-1)^n\left(\frac{\pi}{2}-\arctan n\right)$

203. $\sum_{n=1}^{\infty}(-1)^n\frac{1}{\sqrt{n}}$

14.7.6 Power Series

An infinite series of the form:

$$\sum_{K=0}^{\infty}a_K(x-c)^K=a_0+a_1(x-c)+a_2(x-c)^2+\cdots$$

is called a power series in x.

For each number substituted for x we obtain a series of constant terms which may converge or diverge. Let I be the set of numbers for which the power series converges. Then we can define a function f; $f:I\to\mathbb{R}$ by the rule for every $x\in I$, $f(x)=\sum_{K=0}^{\infty}a_K(x-c)^K$.

The domain I of f is called the *interval* of convergence of $\sum_{K=0}^{\infty}a_K(x-c)^K$ and *always* has one of the following forms:

(i) I is an interval centred at c with endpoints $c-r$ and $c+r$ where $r>0$. The number r is called the radius of convergence of the power series. The endpoints, $c-r$ and $c+r$, may or may not be in the domain I.

(ii) $I = (-\infty, \infty)$.

(iii) $I = (c)$.

Examples

1. $\sum\limits_{K=0}^{\infty} \frac{(x+4)^K}{3^K}$. Choose a fixed but unspecified number x_0 and apply the ratio test.

$$\left|\frac{a_{K+1}}{a_K}\right| = \frac{\left|\frac{(x_0+4)^{K+1}}{3^{K+1}}\right|}{\left|\frac{(x_0+4)^K}{3^K}\right|} = \frac{3^K}{3^{K+1}}\left|\frac{(x_0+4)^{K+1}}{(x_0+3)^K}\right|$$

$$\lim_{K\to\infty}\left|\frac{a_{K+1}}{a_K}\right| = \lim_{K\to\infty}\frac{1}{3}|x_0+4| = \frac{1}{3}|x_0+4|.$$

If $\frac{1}{3}|x_0+4| < 1$ then the series converges by the ratio test.

$$\frac{1}{3}|x_0+4| < 1 \Longrightarrow |x_0+4| < 3 \Longrightarrow x_0 \in (-7,-1).$$

If $\frac{1}{3}|x_0+4| = 1$; i.e. $x_0 = -1$ or minus 7 the ratio test gives no information. We must test these two cases separately as follows:

(a) Substitute $x_0 = -1$ in the given power series to obtain $\sum\limits_{K=0}^{\infty}\left(\frac{3}{3}\right)^K$.

This is a divergence geometric series.

(b) For $x_0 = -7$ we again get a divergent series: $\sum\limits_{K=0}^{\infty}(-1)^K$.

Therefore the interval of convergence is (minus 7, minus 1) centred at $x =$ minus 4 with radius 3.

2. $\sum\limits_{K=0}^{\infty} K!(x-a)^K$. For a fixed number x, $x \neq a$ we apply the ratio test to obtain

$$\lim_{K\to\infty}\left|\frac{(K+1)!(x-a)^{K+1}}{K!(x-a)^K}\right| = \lim_{K\to\infty}(K+1)|x-a| = \infty,\ x \neq a.$$

Therefore the series diverges for $x \neq a$. At $x = a$ the series contains only one non zero term. To see this write $f(x) = \sum\limits_{K=0}^{\infty} K!(x-a)^K$ and expand the right hand side and evaluate $f(a)$:

$$\begin{aligned} f(x) &= 1 + (x-a) + 2!(x-a)^2 + 3!(x-a)^3 + \cdots \\ f(a) &= 1. \end{aligned}$$

3. $\sum\limits_{K=0}^{\infty} \frac{(x-c)^K}{K!}$. For $x \neq a$ the ratio test gives:

$$\lim_{K\to\infty}\left|\frac{(x-c)^{K+1}}{(K+1)!}\cdot\frac{K!}{(x-c)^K}\right| = \lim_{K\to\infty}\frac{|x-c|}{K+1} = 0.$$

Note this limit does not depend on what number is substituted or x. Therefore the interval of convergence is $(-\infty, \infty)$.

PROBLEMS

Find the radius of convergence, test the series at the endpoints, and give the interval of convergence for the following power series.

204. $\sum_{K=0}^{\infty}(-1)^K x^K$

205. $\sum_{K=1}^{\infty}\frac{x^K}{2K-1}$

206. $\sum_{K=1}^{\infty}\frac{(-1)^K 2^K x^K}{3K+1}$

207. $\sum_{K=0}^{\infty}\frac{x^K}{K!}$

208. $\sum_{K=1}^{\infty}\frac{(x+2)^{K-1}}{K^2}$

209. $\sum_{K=0}^{\infty}\frac{(-1)^K x^K}{(2K+1)^2 3^{K+1}}$

210. $\sum_{K=1}^{\infty}\frac{(K+1)x^{2K}}{5^K}$

211. $\sum_{K=0}^{\infty}(-1)^K K! x^K$

212. $\sum_{K=0}^{\infty}\frac{K^2(x+2)^K}{K+1}$

213. $\sum_{K=0}^{\infty}\frac{(x-c)^{2K}}{(2K)!}$

214. $\sum_{K=1}^{\infty}\frac{(x+1)^K}{K\sqrt{K+1}}$

215. $\sum_{K=1}^{\infty}\frac{K!}{K^K}x^K$

216. $\sum_{K=1}^{\infty}\frac{1}{K}\left(\frac{x}{4}-1\right)^K$

217. $\sum_{K=0}^{\infty}\frac{1}{3K-1}\left(\frac{x}{3}+\frac{2}{3}\right)^K$

218. $\sum_{K=1}^{\infty}\frac{Kx^{2K+3}}{(K+1)^2}$

219. $\sum_{K=1}^{\infty}\frac{(x-3)^{4K}}{K^{\frac{1}{K}}}$

220. $\sum_{K=2}^{\infty}\frac{x^K}{\ln K}$

211. $\sum_{K=0}^{\infty}\frac{10^K}{K!}x^K$

222. $\sum_{K=2}^{\infty}\frac{(-1)^K x^K}{\ln K}$

223. $\sum_{K=0}^{\infty}K^2 x^K$

14.7.7 Properties of Functions Represented by Power Series

Let f be a function defined by a power series

$$f(x)=\sum_{K=0}^{\infty}a_K(x-c)^K$$

with a nonzero radius of convergence r or ∞ (we say f with domain $(c-r,c+r)$ or $(-\infty,\infty)$ has a power series representation). Then:

1. The function f is continuous on the open interval $(c-r,c+r)$.
2. $f'(x)=\sum_{K=0}^{\infty}\frac{d}{dx}\left[a_K(x-c)^K\right]=\sum_{K=0}^{\infty}Ka_K(x-c)^{K-1}$.
   This series has the same radius of convergence as the series defining f.
3. $\int f(x)dx=\int\left[\sum_{K=0}^{\infty}a_K(x-c)^K\right]dx=\sum_{K=0}^{\infty}\frac{a_K}{K+1}(x-c)^{K+1}+C.$
   This series has the same radius of convergence as the series defining f.

4. For $|b-a| < r_1$

$$\int_a^b f(x)dx = \sum_{K=0}^{\infty} \int_a^b a_K(x-c)^K dx = \sum_{K=0}^{\infty} \frac{a_K}{K+1}(b-a)^{K+1}.$$

5. Let $\sum\limits_{K=0}^{\infty} a_K(x-c)^K$ be a power series with a radius of convergence r. Define $f(x)$ as follows:

$$f(x) = \sum_{K=0}^{\infty} a_K(x-c)^K + \sum_{K=n+1}^{\infty} a_K(x-c)^K.$$

For each x in the domain of f we obtain *an approximation* of $f(x)$; the nth partial sum of the series. We can write

$$f(x) = \sum_{K=0}^{n} a_K(x-c)^K + E_n(x)$$

where

$$E_N(x) = f(x) - \sum_{K=0}^{n} a_K(x-c)^K = \sum_{n+1}^{\infty} a_K(x-c)^K$$

is the error term.

$E_N(x)$ has the following property: $\lim\limits_{n\to\infty} E_n(x) = 0.$

Therefore we cannot only approximate $f(x)$ by a polynomial $\sum\limits_{K=0}^{n} a_K(x-c)^K$ but control the size of the error by our choice of n.

6. If a function, f, is represented by a power series; that is,

$$f(x) = \sum_{K=0}^{\infty} a_K(x-c)^K$$

for all x in an open interval containing c, then

$$f(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n + \cdots$$

This form is called Taylor series for $f(x)$ at c.

7. If a function, f, has derivatives of all orders in an interval containing c and if $\lim_{n\to\infty} R_n(x) = 0$.

($R_n(x)$ is the error term in Taylor's Theorem) for every x in the given interval, then $f(x)$ is represented by the Taylor series for $f(x)$ at c.

Examples

1. Find $f'(x)$ where $f(x) = 1 + 2x + 3x^2 + \cdots + (K+1)x^K + \cdots$

   Solution: $f'(x) = 0 + 2 + 6x + 12x^2 + \cdots + K(K+1)x^{K-1} + \cdots.$

   Since the radius of convergence of $f(x) = 1$, the domain of f and f' is $(-1,1)$.

2. Find $\int f(x)dx$ where $f(x) = 1 + 2x + 3x^2 + 4x^3 + \cdots$ and express $f(x)$ in closed form.

Solution: $f(x) = \sum_{K=0}^{\infty} (K+1)x^K$.

Thus

$$\int f(x)dx = \sum_{K=0}^{\infty} \int (K+1)x^K dx = \sum_{K=0}^{\infty} x^{K+1} + C.$$

Now you should recall that $\sum_{K=0}^{\infty} x^K = \frac{1}{1-x}$, $|x| < 1$. Since

$$\sum_{K=0}^{\infty} x^K = 1 + \sum_{K=0}^{\infty} x^{K+1} + C,$$

$$\int f(x)dx = \frac{1}{1-x} + C$$

and

$$f(x) = \left(\frac{1}{(1-x)}\right)^2 = \frac{1}{(1-x)^2}$$

on the domain $|x| < 1$.

3. Represent $f(x) = \frac{1}{1+x}$ as a power series.

Solution:

We know $g(x) = \sum_{K=0}^{\infty} x^K = \frac{1}{1-x}$ for $|x| < 1$ (The sum of the geometric series $1 + x + x^2 + x^3 + \cdots$). The solution set of $|x| < 1$ is $x \in (-1, 1)$. Thus for $-x \in (-1, 1)$ we can substitute $-x$ for x in $g(x)$ to obtain:

$$g(-x) = \sum_{K=0}^{\infty} (-x)^K = \frac{1}{1+x} = f(x).$$

Now $|-x| < 1$ is the same as $|x| < 1$, so $f(x) = 1 - x + x^2 - x^3 + \cdots$ on the interval $|x| < 1$.

Problems

In question 224 to 234 use the result $\sum_{K=0}^{\infty} x^K = \frac{1}{1-x}$, $|x| < 1$ to obtain a power series representation of the given function. For each problem specify the domain of the power series representation. Recall that from a given function we can construct new functions by substitution, differentiation and integrations. The series representation of these new functions can be determined by use of properties listed on page 219.

224. $\frac{1}{(1-x)^2}$

225. $\ln(1+x)$

226. $\frac{1}{1+x^2}$

227. $\arctan x$

228. $\frac{1}{5-x}$

229. $\frac{2}{x^2-4x+3}$

230. $\frac{1}{1-x^4}$

231. $\frac{x}{1-x^4}$

232. $\frac{x}{1-x^2}$

233. $\ln \frac{1+x}{1-x}$

234. $\int_0^x \ln(1-t)dt$