Contents

Chapter 3

Techniques of Differentiation

3.1 Differentiating Functions Involving the Power Rule, Constants and the Sums of Differentiable Functions

For questions numbered 1–23, compute the indicated derivative. The letters $a, b, c, \alpha, \beta, \gamma$, represent constants. Answers should be **simplified** and in a **factored** form wherever possible.

1. $y = 3x^5 - 2x^3$; $\quad D_x y$ $\qquad \left[D_x y = 3x^2(5x^2 - 2)\right]$
2. $z = 3ax^2 + 2bx + c$; $\quad \frac{dz}{dx}$
3. $y = (\sqrt{2})^3 - (\sqrt{2})^5$; $\quad y'$
4. $y = \frac{ax+b}{c}$; $\quad y'$ $\qquad \left[y' = \frac{a}{c}\right]$
5. $y = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!}$; $\quad y'$
6. $y = \left(\frac{7}{2}x^3 - 3x^2 + \frac{1}{3}x - 1\right) - \left(2x^4 + 2x^3 - \frac{1}{2}x^2 - \sqrt{2}\right)$; $\quad D_x y$
7. $G(t) = \alpha + (\alpha + \beta)t + (\alpha + \beta + \gamma)t^2$; $\quad G'(2)$ $\qquad [G'(2) = 5\alpha + 5\beta + 4\gamma]$
8. $y(x) = \frac{ax^2+bx}{x}$; $\quad y'(x)$
9. $F(t) = \frac{a^2+2abt+b^2t^2}{a^2+2ab+b^2}$; $\quad F'(1)$
10. $y = \frac{a^2+2abx+b^2x^2}{a+bx}$; $\quad y'$ $\qquad [y' = b]$
11. $y = \frac{x^2-1}{x+1}$; $\quad y'$
12. $y = \frac{ax^{\frac{1}{2}}+bx^{\frac{3}{2}}+cx^{\frac{5}{2}}}{\sqrt{x}}$; $\quad y'$
13. $f(x) = 5x^{101} + 17x^2 + 8$; $\quad f'(x)$ $\qquad [f'(x) = 505x^{100} + 34x]$
14. $y = (a + bx^2)^3$; $\quad D_x y$
15. $y = \frac{x^3-a^3}{x-a}$; $\quad D_x y$
16. $f(x) = |x|$; $\quad f'(2)$ $\qquad [f'(2) = 1]$
17. $f(x) = x + |x - 3|$; $\quad f'(2)$

18. $f(x) = \frac{|x|}{x}$; $f'(1)$ and $f'(-1)$
19. $f(x) = x + |x^2 - 2|$; $f'(1)$ $[f'(1) = -1]$
20. $f(x) = x + |x^2 - 2|$; $f'(3)$
21. It is true that

    $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \cdots$

    $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + \cdots$

    These are infinite series representations of sin and cos. By differentiating the series term by term, find formulae for $D_x \sin x$ and $D_x \cos x$.
22. $F(x^{\frac{1}{2}}) = ax^{\frac{3}{2}} + bx^{\frac{5}{2}} + cx^5$; $D_{x^{\frac{1}{2}}} F(x^{\frac{1}{2}})$ $\left[D_{x^{\frac{1}{2}}} F(x^{\frac{1}{2}}) = 3ax + 5bx^2 + 10cx^{\frac{9}{2}}\right]$
23. (a) $F(x^2) = x^6 + 3x^2 + 1$; $D_{x^2} F(x^2)$

    (b) $F(\sqrt{x}) = x^2 + x + 2$; $D_{\sqrt{x}} F(\sqrt{x})$

    (c) $F(x+1) = x^2 + 2x + 2$; $D_{(x+1)} F(x+1)$

    (d) $F(a-x) = a^2 - 2ax + x^2 - \sqrt{a-x}$; $D_{(a-x)} F(a-x)$
24. Use the definition of the derivative to prove that $D_x\big(af(x) - bg(x)\big) = aD_x f(x) - bD_x g(x)$.

3.2 Differentiating Functions Involving Products, Powers and Quotients

For questions numbered 25–53, compute the derivative. Answers should be **simplified** and in a **factored** form wherever possible.

25. $y = (a + bx^2)^3$; y' $\left[y' = 6bx(a + bx^2)^2\right]$
26. $y = (x^2 + 2x)(3x + 1)$; $\frac{dy}{dx}$
27. $y = (x^3 + 6x^2 - 2x + 1)(x^2 + 3x - 5)$; y'
28. $y = (3 + x^3)(4 + x^2)$; $D_x y$ $\left[D_x y = x(5x^3 + 12x + 16)\right]$
29. $z = (a - x)(a^2 + ax + x^2)$; $\frac{dz}{dx}$
30. $W = (a + bx)(a^4 - a^3bx + a^2b^2x^2 - ab^3x^3 + b^4x^4)$; W'
31. $f(x) = \frac{a-x}{x}$; $f'(x)$ $\left[f'(x) = -\frac{a}{x^2}\right]$
32. $f(x) = \frac{x}{a-x}$; $f(x)$
33. $y = \frac{x-a}{x+a}$; $D_x y$
34. $y = \frac{x+a}{x-a}$; $D_x y$ $\left[D_x y = \frac{-29}{(x-9)^2}\right]$
35. $F(t) = \frac{1+t}{1+t^2}$; $F'(t)$

36. $F(t) = \frac{t^2+2t+1}{(t+1)^2}$; $F'(t)$
37. $f(x) = \frac{2x^2-3x}{x^2+3}$; $f'(x)$ $\left[f'(x) = \frac{3x^2+12x-9}{(x^2+3)^2}\right]$
38. $T(t^3 - 3t + 3)^3$; $\frac{dT}{dt}$
39. $y = \frac{2x^2-3x+4}{x}$; y'
40. $y = \frac{3x-2}{2x+3}$; y'
41. $\phi = \frac{2\theta+3}{3\theta+2}$; ϕ'
42. $y = \frac{x}{x^2+1}$; $D_x y$
43. $\phi = (2x^3 + x^2 - 4x - 1)^4$; ϕ' $\left[\phi' = 8(3x^2 + x - 2)(2x^3 + x^2 - 4x - 1)^3\right]$
44. $\psi(\phi) = \frac{(\theta+3)^2}{\theta+2}$; $\psi'(\phi)$
45. $\tau(\theta) = (1 + \theta^2)(1 + \theta^3)$; $\tau'(\theta)$
46. $y = \frac{1+x^2}{1-x^2}$; y' $\left[y' = \frac{4x}{(1-x^2)^2}\right]$
47. $y = \frac{1}{x(x+1)}$; y'
48. $y = (x - 1)(x - 2)(x - 3)$; $D_x y$
49. $y = (x - 1)^2(x - 2)(x - 3)$; $D_x y$ $\left[D_x y = (x - 1)(4x^2 - 17x + 17)\right]$
50. $y = (x - 1)^3(x - 2)(x - 3)$; $D_x y$
51. $y = \frac{x+1}{2x+3}(2x - 5)$; y'
52. $y = (x - 1)^3(x - 2)^2(x - 3)$; y' $\left[y' = (x - 1)^2(x - 2)(6x^2 - 26x + 26)\right]$
53. $y = \frac{x+1}{x^2+2x+2}$; $D_x y$
54. Given that $f(x) = u(x) \cdot v(x) \cdot w(x)$, find a formula for $f'(x)$. (*Hint:* Apply the rule for finding the derivative of the product twice.) Apply this formula to differentiate the following functions.
55. (a) $f(x) = (x^2 + 2x - 6)(3x - 2)(x^2 + 5)$

    (b) $f(x) = (3x^2 + x)(x^2 - 3)(4x + 1)$

    (c) $f(x) = (2x + 3)^2(x^2 + 1)$

    (d) $f(x) = (x^2 + x + 1)^3$
56. Given that $f(x) = \frac{u(x)}{v(x)}w(x)$, find a formula for $f'(x)$. (*Hint:* Apply the rule for differentiating a quotient and the rule for differentiating a product.) $\left[f'(x) = \frac{uvw'+u'vw-uv'w}{v^2}\right]$
    Apply this formula to differentiate the following functions.

    (a) $f(x) = \left(\frac{x+2}{3x+1}\right)(x - 6)$

    (b) $f(x) = \left(\frac{x^2-1}{2x+6}\right)(x^2 + 5)$

    (c) $f(x) = \left(\frac{x^3+1}{x^2-3}\right)(x^4 - 2x^3 + 1)$

57. Using the product rule find a formula for the derivative of $f(x) = [u(x)]^2$.
Apply this formula to differentiate the following functions.

    (a) $f(x) = (x^2 + 2x - 1)^2$

    (b) $f(x) = (x^3 + 7x^2 - 8x - 6)^2$

    (c) $f(x) = (x^7 - 2x + 3x^{-2})^2$

58. Given that n is a positive integer and $f(x) = x^{-n}$. Find a formula for $f'(x)$. (*Hint:* Use the quotient rule for differentiating.)
Apply this formula to differentiate the following functions.

    (a) $f(x) = x^2 + 2x - \frac{1}{x^2}$

    (b) $f(x) = \frac{3}{4x^7}$

    (c) $f(\theta) = 3\theta^{-2} - 2\theta^{-3}$

    (d) $f(\phi) = \frac{1}{\phi} - \frac{1}{\phi^2}$

    (e) $y = a + \frac{b}{x}$

59. $\phi(\theta) = \sqrt{3}\,\theta^3 + \frac{3}{\theta^2}$; $\phi'\sqrt{3}$ $\left[\phi'(\sqrt{3}) = \frac{25}{\sqrt{3}}\right]$

60. $f(x) = (x^2 - 3x + 2)(x^3 + 2x^2 - 6x)$; $f'(3)$ and $f'(2)$

In problems 61 to 64, find the equation of the tangent line to the graph of $y = f(x)$ at the indicated point.

61. $y = a + \frac{b}{x}$; $\left(a, a + \frac{b}{a}\right)$

62. $y = \frac{ax+b}{c}$; $\left(\frac{-b}{a}, 0\right)$ $\left[y = \frac{ax+b}{c}\right]$

63. $y = 3x^{-2} - 2x^{-3}$; $\left(2, \frac{1}{2}\right)$

64. $y = 3x^{-2} - 2x^{-3}$; $\left(\frac{1}{2}, -4\right)$

65. Find the point, in the first quadrant, on the parabola $y = x^2 + 4$, such that the tangent line at that point passes through the origin. $[(2, 8)]$

66. If $f(x) = \sum_{k=0}^{30} \left(x^k - x^{k+1}\right)$, find $f'(x)$.

3.3 Differentiating the Composite of Functions Using the Chain Rule

Differentiate the following functions and **simplify** the answer.

67. $(3x + 5)^{10}$; $D_x y$

68. $F(x) = (4x^3 + 3x - 1)^7$; $F'(x)$ $\left[F'(x) = 7(4x^3 + 3x - 1)^6(12x^2 + 3)\right]$

69. $(6 - 3x)^7$; $D_x y$

70. $F(x) = (3x^2+4)^5; \quad F'(x)$
71. $(x+5)^{-3}; \quad D_x y$ $\left[D_x y = -3(x+5)^{-4}\right]$
72. $f(x) = (x^2+3x-2)^4; \quad \frac{df(x)}{dx}$
73. $(x^3+2x-3+x^{-2})^4 = y; \quad D_x y$
74. $y = \frac{1}{x^3+3x^2-6x+4}; \quad D_x y$ $\left[D_x y = \frac{-3(x^2+3x-2)}{(x^3+3x^2-6x+4)^2}\right]$
75. $(x^2+2-x^{-2})^{-1} = y; \quad D_x y$
76. $y = (x^2-3x+2)^{-5}; \quad D_x y$
77. $(x^2+1)^2(x^3-2x)^2 = y; \quad D_x y$ $\left[D_x y = 2x(x^2+1)(x^2-2)(5x^4-3x^2-2)\right]$
78. $f(x) = (x^3+2x-6)^3(x^2-4x+5)^7; \quad f'(x)$
79. $(x^2-x^{-1}+1)(x^3+2x-6)^7 = y; \quad D_x y$
80. $f(x) = \frac{(3x-6)^{-1}}{(x+3)^{-2}}; \quad f'(x)$ $\left[f'(x) = \frac{3(x^2+3)(x^2-4x-1)}{2(x-3)^2}\right]$
81. $\frac{(x^2+1)^3}{(x^2+2)^2} = y; \quad D_x y$
82. $y = \left(\frac{3x-2}{2x+1}\right)^7; \quad D_x y$
83. $\frac{(x^2+2)^2}{(x^2+x^{-1})^3} = y; \quad D_x y$ $\left[D_x y = -\frac{(x^2+2)(2x^5+12x^3-7x^2-6)}{x^2(x^2+x^1)^4}\right]$
84. $\phi = (2x^3+x^2-4x-1)^4; \quad \phi'$
85. $\frac{(x^{-1}+x^2)^{-1}}{(x^3-2x^{-2})^{-2}} = y; \quad D_x y$
86. $y = \frac{1}{x(x+1)}; \quad D_x y$ $\left[D_x y = -\frac{2x+1}{x^2(x+1)}\right]$
87. $(x+5)^2(3x-6)^3(7x^2+1)^4 = y; \quad D_x y$
88. $\phi(t) = \frac{2t(t-3)^3-3t^2(t-2)^2}{(t-2)^6} = \phi'(t)$
89. $\frac{(x^2+1)^5(3x-7)^8}{(x^2+5x-4)^6} = y; \quad D_x y$
$\left[D_x y = 2(x^2+1)^4(3x-7)^7(x^2+5x-4)^{-7} \cdot (9x^4+97x^3-184x^2+197x+57)\right]$
90. $f(x) = (x^2+2x-3)^{16}(2x+5)^{13}; \quad \frac{df(x)}{dx}$
91. $\frac{(x^{-2}+3x^{-4}+7x^{-5})^{-8}}{(x^2+x^{-2})^{-4}(x^{-1}+x^{-2})^{-3}} = y; \quad D_x y$

3.4 Differentiating Rational Powers of a Function

Differentiate functions numbered 92 to 131, and **simplify** the answer where possible.

92. $x^{\frac{5}{3}} - 3x^{\frac{2}{3}} + 4x^{-\frac{1}{3}} = y; \quad D_x y$ $\left[D_x y = \frac{5}{3}x^{\frac{2}{3}} - 2x^{-\frac{1}{3}} - \frac{4}{3}x^{-\frac{4}{3}}\right]$
93. $f(t) = \sqrt[3]{t^3+3t+1}; \quad f'(t)$
94. $x^{-\frac{2}{3}} - x^{-\frac{3}{4}} + 2x^{\frac{4}{7}} = y; \quad D_x y$

95. $y = (x+1)^3(2x-1)^{\frac{4}{3}};\quad D_xy$ $\left[D_xy = \frac{1}{3}(x+1)^2(2x-1)^{\frac{1}{3}}(26x-1)\right]$

96. $\frac{3}{2}x^{\frac{2}{3}} + 2x^{\frac{1}{2}} - x^{-1} = y;\quad D_xy$

97. $f(x) = \frac{2}{\sqrt{s^2-1}};\quad \frac{df(x)}{ds}$

98. $\frac{x^{\frac{3}{2}} - 2x^{\frac{1}{2}} + 4x^{-\frac{1}{2}}}{5} = y;\quad D_xy$ $\left[D_xy = \frac{3}{10}x^{\frac{1}{2}} - \frac{1}{5}x^{\frac{1}{2}} - \frac{2}{5}x^{-\frac{3}{2}}\right]$

99. $y = \left(3 + \frac{3}{2}x - \frac{1}{4}x^2\right);\quad y'$

100. $2x\sqrt{x} + 3\sqrt[3]{x^2} - 5x\sqrt[5]{x^2} = y;\quad D_xy$

101. $y = x\sqrt{a^2-x^2};\quad \frac{dy}{dx}$ $\left[\frac{dy}{dx} = \frac{(a^2-2x^2)}{\sqrt{a^2-x^2}}\right]$

102. $\frac{x^2-3x+2}{2\sqrt{x}} = y;\quad D_xy$

103. $y = \frac{x^3}{(x^2-1)^{\frac{3}{2}}};\quad D_xy$

104. $(2x+3)^{\frac{10}{3}} = y;\quad D_xy$ $\left[D_xy = \frac{20}{3}(2x+3)^{\frac{7}{3}}\right]$

105. $f(x) = \frac{x}{1-x} - \frac{1}{1-x};\quad f'(x)$

106. $(x^2+2x+3)^{\frac{3}{2}} = y;\quad D_xy$

107. $y = \frac{x}{(a^2-x^2)^{\frac{3}{2}}};\quad D_xy$ $\left[D_xy = \frac{a^2+2x^2}{(a^2-x^2)^{\frac{5}{2}}}\right]$

108. $(2x^3-3x^2+3x-1)^{-\frac{1}{3}} = y;\quad D_xy$

109. $f(x) = (x+1)\sqrt{x^2-2x+2};\quad D_xf(x)$

110. $\sqrt[3]{(x^3+3x+2)^2} = y;\quad D_xy$ $\left[D_xy = 2(x^2+1)(x^3+3x+2)^{-\frac{1}{3}}\right]$

111. $y = (x^3+3x)^{\frac{5}{3}};\quad D_xy$

112. $\frac{(x^2+x-3)\sqrt{x^2+x-3}}{5} = y;\quad D_xy$

113. $H(t) = t\sqrt{9t^2-1};\quad H'(t)$ $\left[H'(t) = \frac{18t^2-1}{\sqrt{9t^2-1}}\right]$

114. $(2x+3)^4(3x-2)^{\frac{7}{3}} = y;\quad D_xy$

115. $F(x) = (a^{\frac{2}{3}} - x^{\frac{2}{3}})^{\frac{3}{2}};\quad F'(x)$

116. $(2x-1)^{\frac{5}{2}}(7x-3)^{\frac{3}{7}} = y$ $\left[D_xy = (2x-1)^{\frac{3}{2}}(7x-3)^{\frac{4}{7}}(41x-18)\right]$

117. $y = \sqrt{1+2\sqrt{x}};\quad y'$

118. $\sqrt[3]{(x+1)^2x-1} = y;\quad D_xy$

119. $\lambda(x) = (2+x)\sqrt{2-x};\quad D_x\lambda(x)$ $\left[D_x\lambda(x) = \frac{2-3x}{2\sqrt{2-x}}\right]$

120. $G(t) = (\sqrt{a} - \sqrt{t})^2;\quad G'(t)$

121. $f(t) = \sqrt{\frac{1+t^2}{1-t^2}};\quad f'(t)$

122. $f(t) = \sqrt{\frac{1-t^2}{1+t^2}}$; $f'(t)$ $\left[f'(t) = \frac{-2t}{(1+t^2)^{\frac{3}{2}}(1-t^2)^{\frac{1}{2}}}\right]$

123. $y = \frac{x^2}{(1-x^2)^{\frac{1}{2}}}$; $D_x y$

124. $y = (4x^3 - 3x^2 - \pi)^{\frac{3}{2}}$; $D_x y$

125. $f(r) = (1+r)^2(r-1)^{-\frac{1}{2}}$; $f'(r)$ $\left[f'(r) = \frac{(3r-5)(r+1)}{2(r-1)^{\frac{3}{2}}}\right]$

126. $\phi(x) = \sqrt{x-1}\,\sqrt{x+1}$; $D_x\phi(x)$

127. $y = \sqrt{\frac{1-x}{1+x}}$; $D_x y$

128. $y = \sqrt{2x} + \frac{1}{\sqrt{2x}}$; $D_x y$ $\left[D_x y = \frac{2x-1}{(2x)^{\frac{3}{2}}}\right]$

129. $y = \sqrt{2x} + \sqrt{\frac{2}{x}}$; $D_x y$

130. $y = \sqrt{\frac{x}{2}} + \sqrt{\frac{2}{x}}$; $D_x y$

131. $y = \frac{1}{x+\sqrt{1+x}}$; $D_x y$ $\left[D_x y = \frac{x}{\sqrt{1+x^2}} - 1\right]$

3.5 Derivatives of Trigonometric Functions

Differentiate the following functions:

132. $y = \cos 2x$

133. $y = \cos x \sin x$

134. $y = \cos 2x \sin x$ $[D_x y = \cos^3 x - \sin 2x \sin x]$

135. $\phi = \sec 3\theta$

136. $r = a\sqrt{\cos 2\theta}$

137. $\lambda(\theta) = \sin^2 3\theta + \cos^2 3\theta$ $[d\lambda/d\theta = 0]$

138. $x = a(\theta - \sin\theta)$

139. $\tau(\theta) = \tan\sqrt{1-\theta}$

140. $f(\theta) = \cot(a\theta^2 + 5)$ $[f'(\theta) = -2a\theta\csc^2(a\theta^2+5)]$

141. $F(x) = A\csc^2 3x$

142. $\mu(\theta) = (\tan 2\theta) - 2\theta$

143. $y = \frac{1-\tan 2x}{\sec 2x}$ $[y' = -2(\cos 2x + \sin 2x)]$

144. $y = (\csc 2x - \cot 2x)^2$

145. $\phi(t) = \sqrt{\cos 2t + 1}$

146. $y = \sin^4 x + \cos^4 x$ $[y' = -\sin 4x]$

147. $f(x) = \frac{\sin 2x}{x}$

148. $f(x) = x \sin \frac{1}{x}$

149. $f(x) = x^2 \sin \frac{1}{x}$ $\left[f'(x) = -\cos\frac{1}{x} + 2x \sin\frac{1}{x}\right]$

150. $G(\phi) = \cos^3 \phi \sin \phi$

151. $y = \cos^2 \sqrt{x}$

152. $\phi(x) = \frac{\sin 2x}{1+\cos 2x}$ $\left[\phi'(x) = \frac{2}{(1+\cos 2x)} = \sec^2 x\right]$

153. $y = \sqrt{1 + \tan^2 3\theta}; \quad -\frac{\pi}{2} < 3\theta < \frac{\pi}{2}$

154. $a(x) = \tan \sqrt{x}$

155. $\beta(x) = \frac{\sec 2x}{\tan 2x + 1}$ $\left[\beta'(x) = \frac{2(\sin 2x - \cos 2x)}{(\sin 2x + \cos 2x)^2}\right]$

156. $y = \frac{1 - \tan^2 x}{\tan x}$

157. $y = \frac{a}{\cos 4x}$

158. $y = \cos^2 ax$ $[y' = -2a \cos ax \sin ax]$

159. $y = -\sin^2 ax$

160. $y = \cos^2 ax - \sin^2 ax$

161. $\beta(\theta) = \sqrt{\sin\theta}$ $\left[D_\theta \beta(\theta) = \frac{\cos\theta}{(2\sqrt{\sin\theta})}\right]$

162. $\lambda(\theta) = \sqrt{\sin\sqrt{\theta}}$

163. $Z = \sin\left(\frac{1}{x^2}\right)$

164. $f(\theta) = \theta + \cot\theta - \frac{1}{3}\cot^3\theta$ $\left[f'(\theta) = \cot^4\theta\right]$

165. $F(\theta) = \frac{\tan\theta - 1}{\sec\theta}$

166. $y = \cos^2 \sin x$

167. $y = \cos^2 x \sin x$ $\left[D_x y = -2\cos x \sin^2 x + \cos^3 x\right]$

168. $y = \tan x \sec x$

169. $y = \tan \sec x$

170. $y = \sin \sin \cos \tan x^2$ $\left[D_x y = \cos \sin \cos \tan x^2 \cdot \cos \cos \tan x^2 \cdot (-\sin \tan x^2) \cdot \sec^2 x^2 \cdot 2x\right]$

171. Differentiate the function represented on each side of the following identities. Obtain in this way either new identities or familiar or obvious ones.

   (a) $\sin^2\theta + \cos^2\theta = 1$

   (b) $\sec\theta = \frac{1}{\cos\theta}$

   (c) $\tan\theta = \frac{\sin\theta}{\cos\theta}$ $\left[\sec^2\theta = \frac{\cos^2\theta + \sin^2\theta}{\cos^2\theta}\right]$

   (d) $\sin 2\theta = 2\sin\theta\cos\theta$

171. (e) $\sin(\theta + a) = \sin\theta\cos a + \cos\theta\sin a$

(f) $\tan(\theta + a) = \frac{\tan\theta + \tan a}{1 - \tan\theta\tan a}\left[\sec^2(\theta + a) = \frac{\sec^2\theta(1+\tan^2 a)}{(1-\tan\theta\tan a)^2}\right]$

(g) $\sec^2\theta + \csc^2\theta = \csc^2\theta\sec^2\theta$

3.6 Derivatives of Logarithmic and Exponential Functions

Differentiate the following functions and **simplify** your answer where possible.

172. (a) $f(x) = e^{-x}$ (b) $f(x) = 4e^{x+4}$
(c) $f(x) = e^{x^2}$ (d) $f(x) = (e^x)^2$

173. (a) $f(x) = e(1 + e^x)$ (b) $f(x) = e^2 e^x$
(c) $f(x) = e^3 e^{2x}$ (d) $f(x) = e^3 + e^2 + e^{-x}$

174. (a) $f(x) = ae^2$ (b) $f(x) = \frac{e^3}{e^x}$
(c) $f(x) = \frac{e^x}{e^{x^2}}$ (d) $f(x) = \sqrt{e}\, e^{x+2}$

175. (a) $f(x) = e + e^x$ (b) $f(x) = e^{2x} + 2e^x + 1$
(c) $f(x) = (e^x + 1)^2$ (d) $f(x) = \frac{e^{2x} - 1}{e^x + 1}$

176. (a) $f(x) = xe^{-x}$ (b) $f(x) = x + x^2 e^x$
(c) $f(x) = x^2 + e^{x^2}$ (d) $f(x) = \sqrt{x}\, e^{\sqrt{x}}$

177. (a) $f(x) = x^2 e^x - xe^x + 6e^x$ (b) $f(x) = (e^x + 1)^2$
(c) $f(x) = \sqrt{e^x - 1}$ (d) $f(x) = \sqrt{xe^x}$

178. (a) $f(x) = \frac{x}{e^x + 1}$ (b) $f(x) = \frac{e^x}{x^2 + 1}$
(c) $f(x) = \frac{e(x+1)}{e^x - e^{-x}}$ (d) $f(x) = \frac{1}{2}(e^x - e^{-x})$

179. (a) $f(x) = \frac{1}{2}(e^x + e^{-x})$ (b) $f(x) = \frac{e^x + e^{-x}}{e^x - e^{-x}}$
(c) $f(x) = e^x + xe^{-x}$ (d) $f(x) = \frac{e^x}{e^{-x} + 1}$

180. (a) $f(x) = |e^x|$ (b) $f(x) = e^{|x|}$
(c) $f(x) = \sqrt{e^{2x}}$ (d) $f(x) = e^{|x|} - e^x$

181. (a) $f(x) = \sin e^x$ (b) $f(x) = e^x \sin x$
(c) $f(x) = e^{-x}\sin x$ (d) $f(x) = e^x \sin e^x$

182. (a) $f(x) = e^{\sin x}$ (b) $f(x) = e^2 e^{\sin x}$
(c) $f(x) = e^{\tan x}$ (d) $f(x) = \left(\sin\frac{\pi}{2}\right) e^{\sin x}$

183. (a) $f(x) = e^{-x}\cos x$ (b) $f(x) = e^{x+1}\cos(x + \pi)$
(c) $f(x) = e^{\pi}\tan(x + \pi)$ (d) $f(x) = e^{2\pi}\sin(x + 2\pi)$

184. (a) $f(x) = e^{\ln x}$ (b) $f(x) = \ln(e^x)$
(c) $f(x) = \ln(e^{-x})$ (d) $f(x) = e^{\ln\sqrt{x}}$

185. (a) $f(x) = xe^{\ln x^2}$ (b) $f(x) = x^2 e^{\ln\sqrt{x}}$
(c) $f(x) = \ln e^{x^3}$ (d) $f(x) = \frac{x}{\ln(e^{x^2})}$

186. (a) $f(x) = x - e^{\ln x}$ (b) $f(x) = \sin^2 x + \ln(e^{\cos^2 x})$
(c) $f(x) = e^x \ln(e^x)$ (d) $f(x) = x\ln\left(e^{\frac{1}{x}}\right)$

187. (a) $f(x) = e^{x+\ln x}$ (b) $f(x) = \ln(x + e^x)$
(c) $f(x) = e^{\pi}e^{\ln x}$ (d) $f(x) = \ln(ae^x)$

188. (a) $f(x) = \ln x^2$ (b) $f(x) = 2\ln x$
(c) $f(x) = \ln ax$ (d) $f(x) = \ln x - \ln 2x$

189. (a) $f(x) = \ln 5x$ (b) $f(x) = \frac{1}{2}\ln x^2$
(c) $f(x) = \ln(x^2 + x)$ (d) $f(x) = \ln\left(\frac{1}{x}\right)$

190. (a) $f(x) = (\ln x)\ln(x^2 - 1)$ (b) $f(x) = (\ln x)^2$
(c) $f(x) = \ln\sqrt{x}$ (d) $f(x) = \sqrt{\ln x}$

191. (a) $f(x) = \ln(\sin x)$ (b) $f(x) = \ln(\tan x)$
(c) $f(x) = \sin(\ln x)$ (d) $f(x) = \cos^2(\ln x)$

192. (a) $f(x) = \ln(\sin^2 x)$ (b) $f(x) = \ln(\sec x)$
(c) $f(x) = -\ln\cos x$ (d) $f(x) = \ln(\cos x^2)$

193. (a) $f(x) = \sin(\ln x)$ (b) $f(x) = \tan(\ln 2x)$
(c) $f(x) = \sec[\ln(x + 10)]$ (d) $f(x) = \cos(\ln x^2)$

194. (a) $f(x) = \ln\sqrt{\frac{1+\sin x}{1-\sin x}}$ (b) $f(x) = \ln\left(\tan\left(\frac{\pi}{4} + \frac{x}{2}\right)\right)$
(c) $f(x) = \ln\sqrt{1 - 2\sin^2 x}$ (d) $f(x) = \ln(\sec x + \tan x)$

195. (a) $f(x) = \ln(\csc x + \cot x)$ (b) $f(x) = \ln(\cos x \sin x)$
(c) $f(x) = \ln(\sin x) - \ln(\cos x)$ (d) $f(x) = \ln(\cos^2 x - \sin^2 x)$

196. (a) $f(x) = \frac{\ln(x^2+1)}{x}$ (b) $f(x) = \ln\frac{x^2+1}{x^2+3}$
(c) $f(x) = \ln\frac{x^2-1}{x}$ (d) $f(x) = \frac{\ln(x+2)}{\ln(x+1)}$

197. (a) $f(x) = \frac{x}{\ln(x^2-2)}$ (b) $f(x) = \frac{1}{\ln x}$
(c) $f(x) = \frac{\ln(\sin x)}{\ln(\cos x)}$ (d) $f(x) = \frac{x}{(\ln x)^2}$

198. (a) $f(x) = x\ln x - x$ (b) $f(x) = \frac{\ln\sqrt{x}}{\ln(x+e^x)}$
(c) $f(x) = \ln\left(\frac{e^x - e^{-x}}{2}\right)$ (d) $f(x) = \ln(\ln x)$

199. (a) $f(x) = \ln|x|$ (b) $f(x) = \frac{1}{x\ln x}$
(c) $f(x) = \ln\frac{\sqrt{x^2+1}+x}{\sqrt{x^2+1}-x}$ (d) $f(x) = \ln\left(x+\sqrt{x^2-1}\right)$

200. (a) $f(x) = -\ln\left(\sqrt{x^2+1}-x\right)$ (b) $f(x) = \ln\frac{1+\sqrt{x}}{1-\sqrt{x}}$
(c) $f(x) = \ln\sqrt{\frac{1+x}{1-x}}$ (d) $f(x) = \ln\left(\frac{a-x}{a+x}\right)$

201. (a) $f(x) = Ae^{\ln kx}$ (b) $f(x) = Ae^{x\ln k}$
(c) $f(x) = Ae^{\ln k^x}$ (d) $f(x) = A\ln e^{xk}$

202. (a) $f(x) = 10^x$ (b) $f(x) = 10^{x^2-1}$
(c) $f(x) = 4^{x+1}$ (d) $f(x) = 2^{x^2+x}$

203. (a) $f(x) = 2^{\sqrt{x}}$ (b) $f(x) = 2^{2x} + 4^x + 1$
(c) $f(x) = (10^{-x} + 10^x)^2$ (d) $f(x) = 2^{-x} + 2^x$

204. (a) $f(x) = x^2 2^x$ (b) $f(x) = x^{10} + 10^x$
(c) $f(x) = \sqrt{x}\,2^x$ (d) $f(x) = 2^x x^3 - x^2 2^x$

205. (a) $f(x) = \frac{2^x}{\ln 2}$ (b) $f(x) = \frac{1}{2^x\ln 2}$
(c) $f(x) = 10^{\frac{x}{\ln 10}}$ (d) $f(x) = \frac{2^x}{3^x}$

206. (a) $f(x) = \frac{2^a 2^{\frac{x}{2}}}{\ln 2}$ (b) $f(x) = 2^{\log_2 x}$
(c) $f(x) = \log_2 2^x$ (d) $f(x) = 10^{\log_{10} ax}$

207. (a) $f(x) = \frac{\log_2 x}{\log_2 e}$ (b) $f(x) = \ln 2\log_2 x$
(c) $f(x) = \log_{10}(x^{\ln 10})$ (d) $f(x) = \log_{10}(10^{\ln x})$

208. (a) $f(x) = \log_2 8\log_2 x^{\frac{1}{3}}$ (b) $f(x) = \frac{\log_{10}(x+1)}{\log_{10}(x-1)}$
(c) $f(x) = A10^{(\log_{10} 2)x}$ (d) $f(x) = A10^{\log_{10} 2x}$

209. (a) $f(x) = 2^x\log_2 x$ (b) $f(x) = \log_2(x2^x)$
(c) $f(x) = 2^{\log_2 k^x}$ (d) $f(x) = 2^{x\log_2 k}$

210. (a) $f(x) = 2^x + \log_2 2^x$ (b) $f(x) = 2^x + \sin 2^x$
(c) $f(x) = \frac{\sin 10^x}{10^x}$ (d) $f(x) = x\log_{10} x$

3.7 Higher Order Derivatives

In problems 211 to 231 do the following:

(a) Compute the first derivative, **simplify**, and **identity** your answer.

(b) Compute the second derivative, simplify where possible, and identify your answer.

211. (a) $y = 4x^{\frac{3}{2}}$ (b) $y = 9x^{\frac{4}{3}}$
(c) $y = \sqrt[3]{x}$ (d) $y = \frac{1}{x}$

212. (a) $y = x^{-2}$ (b) $y = ax + b$
(c) $y = ax^2 + bx + c$ (d) $y = \sqrt{x}$

213. (a) $y = \sin^2 x$ (b) $y = 2 \sin x \cos x$
(c) $y = \tan(x^2 + 1)$ (d) $y = \sec x^2$

214. (a) $y = \sin^2 x$ (b) $y = \cos^2 x - \sin^2 x$
(c) $y = 1 + \tan^2 x$ (d) $y = \tan x \cot x$

215. (a) $y = \csc x \sin x$ (b) $y = \sin x \cos \frac{\pi}{3} + \cos x \sin \frac{\pi}{3}$
(c) $y = 2\cos^2 x - 1$ (d) $y = 1 - 2\sin^2 x$

216. (a) $y = \frac{1-\cos^2 x}{2}$ (b) $y = \sin^x$
(c) $y = \tan x - 6 \sin x \cos x$ (d) $y = \frac{1+\cos 2x}{2}$

217. (a) $y = \sin x + \cos x$ (b) $y = A \sin \omega t$
(c) $y = E \sin(\omega t + \theta)$ (d) $y = A \cos \omega t + B \sin \omega t$

218. (a) $y = \frac{x}{2(x+1)}$ (b) $y = \frac{-1}{(x+1)^3}$
(c) $y = \frac{x}{x-1}$ (d) $y = \frac{2}{(x-1)^3}$

219. (a) $y = \frac{x}{3-x}$ (b) $y = \frac{3}{(3-x)^2}$
(c) $y = \frac{x-a}{x+a}$ (d) $y = \frac{2a}{(x+a)^2}$

220. (a) $y = \frac{1+x+x^2}{x}$ (b) $y = \left(x + \frac{1}{x}\right)^2$
(c) $y = ((x-1)(x+1))^{\frac{1}{2}}$ (d) $y = \frac{x}{\sqrt{x^2-1}}$

221. (a) $y = \frac{x^5-5x+7}{125}$ (b) $y = \frac{x^4-1}{25}$
(c) $y = \sqrt{2x} + \frac{1}{\sqrt{2x}}$ (d) $y = \frac{1}{x+\sqrt{1+x^2}}$

222. (a) $y = \sqrt{x^2+1}$ (b) $y = \frac{1+x}{1-x}$
(c) $y = (x^3 + 3x + 7)^4$ (d) $y = \frac{x^2}{x^2-4}$

223. (a) $y = \frac{x}{x^2+1}$ (b) $y = \frac{x}{x^3+4}$
(c) $y = \left(\frac{x}{x+1}\right)^5$ (d) $y = \frac{x}{\sqrt{x-1}}$

224. (a) $y = e^{x^2}$ (b) $y = xe^x$
(c) $y = (x+2)e^x$ (d) $y = \frac{e^x}{x}$

225. (a) $y = e^{\frac{1}{x}}$ (b) $y = e^x \sin x$
(c) $y = \frac{1}{2}(e^x - e^{-x})$ (d) $y = \frac{1}{2}(e^x + e^{-x})$

226. (a) $y = \sqrt{x}\, e^{\sqrt{x}}$ (b) $y = e^{\sin x}$
(c) $y = \cos e^{x}$ (d) $y = xe^{-x}$

227. (a) $y = \ln(x^2 - x)$ (b) $y = \ln x^2$
(c) $y = \ln x - 2\ln x$ (d) $y = \ln^2 x$

228. (a) $y = x \ln x$ (b) $y = (x+3)\ln x$
(c) $y = \ln x^x$ (d) $y = \frac{\ln x}{x}$

229. (a) $y = x \ln x^2$ (b) $y = \ln\left(\frac{x}{x-1}\right)$
(c) $y = \ln(\ln x)$ (d) $y = \ln(\ln^3 x)$

230. (a) $y = e^{x + \ln x}$ (b) $y = \log_a b^x$
(c) $y = \ln\sqrt{\frac{1+x}{1-x}}$ (d) $y = \ln\left(\frac{1+\sqrt{x}}{1-\sqrt{x}}\right)$

231. (a) $y = e^{-\ln x^2}$ (b) $y = e^{\ln\left(\frac{1}{x}\right)}$
(c) $y = 2^x + x^2$ (d) $y = \ln\left(\frac{1}{x}\right)$

In problems 232 to 253 find the indicated derivative.

232. (a) $f'''(x)$ where $f(x) = x^3(2x-1)^4$
(b) $f'''(x)$ where $f(x) = \frac{1-x}{1+x}$

233. (a) $f^{(5)}(x)$ where $f(x) = ax^4 + bx + c$
(b) $f'''(x)$ where $f(x) = (x^2+1)^3$

234. (a) $f'''(x)$ where $f(x) = \sqrt{x-1}$
(b) $f'''(x)$ where $f(x) = x(x+1)^3$

235. (a) $f'''(x)$ where $f(x) = \frac{(1+x)^2}{x^2}$
(b) $f^{(4)}(x)$ where $f(x) = x^4 - 2x^{\frac{3}{2}} + \frac{3}{x}$

236. (a) $f'''(x)$ where $f(x) = x^{\frac{5}{2}} - 2x^{\frac{2}{3}} + x$
(b) $f'''(x)$ where $f(x) = (2x+1)^{\frac{5}{2}}$

237. (a) $f'''(x)$ where $f(x) = \frac{x}{x-1}$
(b) $f'''(x)$ where $f(x) = \ln(1+ax)$

238. (a) $f^{(4)}(x)$ where $f(x) = A\sin\omega x$
(b) $f^{(3)}(x)$ where $f(x) = \ln\left(\frac{1}{x}\right)$

239. (a) $f^{(5)}(x)$ where $f(x) = xe^{x}$
(b) $f^{(5)}(x)$ where $f(x) = e^{x^2}$

240. (a) $f''(1)$ where $f(x) = x^3h(x),\ h(1) = 3,\ h'(1) = \frac{1}{2}$ and $h''(1) = 4$

(b) $f'''(5)$ where $f(x) = (x-1)^{\frac{1}{2}}g(x),\ g(5) = -1,\ g'(5) = \frac{1}{3},\ b''(5) = 2$ and $g'''(5) = 10$

241. (a) $f''(0)$ where $f(x) = e^x g(a),\ g(0) = 1,\ g'(0) = 0,$ and $g''(0) = -1$

(b) $f'''(x)$ where $f(x) = e^x g(x)$ and $g'(x) = g(x)$

242. (a) $f''(x)$ where $f(x) = \ln(h(x))$ and $h'(x) = h(x)$

(b) $f''(x)$ where $f(x) = xg(x)$ and $g'(x) = g(x)$

243. (a) $(fg)''(0)$ where $f(0) = 2,\ f'(0) = -4,\ f''(0) = \frac{1}{5},\ g(0) = 10,\ g'(0) = 2$ and $g''(0) = -1$

(b) $f'''(1)$ where $f(x) = [h(x)]^3,\ h(1) = 2,\ h'(1) = \frac{1}{3}$ and $h''(1) = \frac{1}{4}$

244. (a) $\phi''(-2)$ where $\phi(x) = \sqrt{1-g(x)},\ g(-2) = -3,\ g'(-2) = 3$ and $g''(-2) = 5$

(b) $f''(1)$ where $f(x) = g(x^3+1),\ g'(2) = 3$ and $g''(2) = \frac{1}{4}$

245. (a) $f'''(9)$ where $f(x) = xg(\sqrt{x}),\ g'(3) = 6,\ g''(3) - 1$ and $g'''(3) = 2$

(b) $(h \circ g)''(0)$ where $g(0) = 1,\ g'(0) = 2,\ g''(0) = \frac{1}{2},\ h'(0) = 0,\ h''(0) = -1,\ h'(1) = 6$ and $h''(1) = 10$

246. (a) $g^{(n)}(x)$ where $g(x) = (x+1)^{-\frac{1}{2}}$

(b) $h^{(n)}(x)$ where $h(x) = (1-2x)^{\frac{2}{3}}$

247. (a) $f^{(n)}(x)$ where $f(x) = x + \frac{1}{x}$

(b) $f^{(n)}(x)$ where $f(x) = x^2 + x + \sqrt{x}$

248. (a) $f^{(n)}(x)$ where $f(x) = x^n$

(b) $f^n(x)$ where $f(x) = e^{-x}$

249. (a) $f^{(n)}(x)$ where $f(x) = \frac{x+1}{x-1}$

(b) $f^{(n)}(x)$ where $f(x) = \sqrt{x}$

250. (a) $f^{(n)}(x)$ where $f(x) = \sqrt{x+1}$

(b) $f^{(n)}(x)$ where $f(x) = \frac{1}{(3-x)}$

251. (a) $f^{(n)}(x)$ where $f(x) = \sqrt{2x+1}$

(b) $f^{(n)}(x)$ where $f(x) = \ln x$

252. (a) $f^n(x)$ where $f(x) = x\ln x$

(b) $f^{(2n)}(\theta)$ where $f(\theta) = \sin a\theta$

253. (a) $F^{(n+1)}(x)$ where $F(x) = x^n \ln x$

(b) $G^n(x)$ where $G(x) = xe^x$

3.8 Implicit Differentiation

Examples:

1. Find $D_x y$ if $\sin y = x + y$.

   Solution: It is not possible to solve for y in terms of x. Therefore differentiate implicitly with respect to x:

   $D_x(\sin y) = D_x(x + y)$

   $\cos y D_x y = 1 + D_x y$ (Chain rule)

   Now solve for $D_x y$:

   $\cos y D_x y - D_x y = 1$

   $D_x y(\cos y - 1) = 1$

   $D_x y = \frac{1}{\cos y - 1}$

2. Find the equation of the tangent line to the graph $x^3 + y^3 - 3x^2y^2 + 1 = 0$ at the point $(1, 1)$.

   Solution: Differentiate implicitly with respect to x :

   $D_x(x^3 + y^3 - 3x^2y^2 + 1) = D_x(0)$

   $3x^2 + 3y^2 D_x y - 6xy^2 - 6x^2 y D_x y = 0$

   (Note here the use of the chain rule and the product rule.)

   Now solve for $D_x y$

   $3y^2 D_x y - 6x^2 y D_x y = -3x^2 + 6xy^2$

   $D_x y(3y^2 - 6x^2 y) = -3x^2 + 6xy^2$

   $D_x y = \frac{-3x^2 - 6xy^2}{3y^2 - 6x^2 y} = \frac{3x(-x+2y^2)}{3y(y-2x^2)}$

   $D_x y = \left(\frac{x}{y}\right)\left(\frac{x-2y^2}{2x^2-y}\right)$

   The slope of the tangent line, m, at $(1, 1)$ is obtained by substituting $x = 1$, $y = 1$ into the expression for $D_x y$ since $m = D_x y$ at $(1, 1)$; the result is $m = -1$. Therefore the equation of the tangent line is $y - 1 = -(x - 1)$ or $y + x - 2 = 0$.

In problems 254 to 263 find y' by implicit differentiation.

254. (a) $2x^2 - 5y^2 = 10$ (b) $x^3 + x^2y + xy^2 + y^3 = 4$
     (c) $x^3 + y^3 - 3axy = 0$ (d) $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$

255. (a) $x^3 + y^3 = 1$ (b) $x^{\frac{1}{3}} - y^{\frac{1}{3}}$
     (c) $xy = 4$ (d) $2xy - y^2 = 1$

256. (a) $(x + y)^2 = (x - y + 1)^2$ (b) $(x^2 - y^2)^2 = x^2 + y^2$
     (c) $\frac{x+y}{x-y} = x^2 + y^2$ (d) $\sqrt{x} - \sqrt{y} = 1$

257. (a) $\frac{x+y}{x-y} = x^2 + y^2$ (b) $x + \sqrt{xy} + y = 4$
     (c) $\frac{x}{y} + \frac{y}{x} = x + y$ (d) $x^2 - xy + y^2 = 1$

258. (a) $\sin(xy) = y$ (b) $\tan y - y = \sin x + x$
     (c) $\sin^2 y = a^2 \cos(2x)$ (d) $\sin y = x^2 + x$

259. (a) $(\sin y)\sin x = 1$ (b) $\sin(x+y) + \frac{1}{2}$
(c) $y = \sin(x+y)$ (d) $\frac{2xy}{\pi} + \sin y$

260. (a) $e^y = \sin x$ (b) $x^2 e^y = 1$
(c) $e^y \sin y = x$ (d) $y + e^y = x$

261. (a) $y^2 e^y = x^2 + 2x$ (b) $e^{3x} = \cos(x+2y)$
(c) $y^2 + e^y = x$ (d) $y^2 e^{\sqrt{y}} = x$

262. (a) $e^{\frac{x}{y}} = y^2$ (b) $e^{3y} = \tan(x + 4y^2)$
(c) $\ln(x-y) = x + y$ (d) $\ln(x+y) = xe^y$

263. (a) $\ln x^2 - \ln 2y = 6$ (b) $x^2 y - \ln y = e^{-2x}$
(c) $e^{2x-y} = 5 + \ln x$ (d) $e^y = \ln x$

3.9 Derivatives of Functions of the Form $f(x) = g(x)^{h(x)}$

Example: Differentiate $y = x^x$

Method I: This method uses the result: $z = e^{\ln z}$

$x^x = e^{\ln(x^x)}$

$(x^x)' = (\ln(x^x))' e^{\ln(x^x)}$ (chain rule)

$(x^x)' = (x \ln x)' x^x$ (property of logarithms and substitution)

$(x^x)' = \left(\frac{x}{x} + \ln x\right) x^x$ (product rule)

$(x^x)' = (1 + \ln x)x^x$

Method II: Take the logarithm of both sides of $y = x^x$ and then differentiate implicitly

$\ln y = \ln(x^x)$

$\ln y = x \ln x$

$\frac{y'}{y} = \left(\frac{x}{x} + \ln x\right)$ (differentiate implicitly)

$y' = y(1 + \ln x)$ (solve for y')

$y' = x^x(1 + \ln x)$ (since $y = x^x$)

Differentiate the following functions using **both** Methods I and Method II.

264. (a) $y = x^{2x}$ (b) $y = x^{(x^2)}$
(c) $y = x^{\sin x}$ (d) $y = x^{\frac{1}{x}}$

265. (a) $y = x^{\cos x}$ (b) $y = x^{\sqrt{x}}$
(c) $y = (\sin x)^x$ (d) $y = x^{\ln x}$

266. (a) $y = x^{\frac{1}{\ln x}}$ (b) $y = x^{(x^x)}$
(c) $y = x^{\sqrt{x-1}}$ (d) $y = x^{(e^x)}$

267. (a) $y = e^{(x^x)}$ (b) $y = (x^x)^x$
(c) $y = \cos x^x$ (d) $y = (x+1)^{x^2}$

268. (a) $y = (\ln x)^{\ln x}$ (b) $y = (\ln x)^{x^2+2}$
(c) $y = (x^2+2)^{\ln x}$ (d) $y = (\cos x)^{\sin x}$

In problems 269 to 273 find the equation of the tangent at the specified point on the curve defined implicitly by the given equation.

269. (a) $x^2 - y^2 = 16; \quad (5, -3)$ (b) $xy = 12; \quad (2, 6)$
(c) $x^3 + y^3 = 7; \quad (-1, 2)$ (d) $xy^2 = 12; \quad (3, -2)$

270. (a) $x^2 + y^2 = 25; \quad (-3, 4)$ (b) $(3x+2y)^{\frac{1}{2}}(x-y) = 8; \quad (4, 2)$
(c) $(x+y)^4 = x - y; \quad (0, -1)$ (d) $\frac{x-y}{x+ey} = -5; \quad (3, -2)$

271. (a) $x\cos y + y\sin x = \frac{\pi}{2}; \quad \left(\frac{\pi}{2}, \pi\right)$ (b) $\sin(xy) + 3y = 4; \quad (\left(\frac{\pi}{2}, 1\right)$
(c) $(\sin x)(\cos y) = \frac{1}{2}; \quad \left(\frac{\pi}{4}, \frac{\pi}{4}\right)$ (d) $\tan xy = 1; \quad \left(\frac{\pi}{4}, 1\right)$

272. (a) $\sec x + \tan y = 1; \quad (0, 0)$ (b) $\csc\left(\frac{\pi x y^2}{x}\right) + \sin\left(\frac{\pi y}{2}\right) + y = 3; \quad (1, 1)$

273. (a) $y^2 + e^y = x; \quad (0, 0)$ (b) $e^{x+y} = 2; \quad (1, -1)$
(c) $x^y = y^x; \quad (1, 1)$ (d) $\ln(x+y) = 1; \quad (0, 3)$

3.10 Logarithmic Differentiation

Differentiate problems 274 to 280 using logarithmic differentiation and simplify your answer where possible.

274. (a) $y = (x+1)(x-1)(1-x)^2$ (b) $y = \left[\frac{(x^2+1)(x+2)}{(x^2+2)(x+3)}\right]^{\frac{1}{2}}$
(c) $y = (x-1)^2(2-x)(3-x)$ (d) $y = \frac{(x+2)^2}{(x+1)(x+3)^3}$

275. (a) $y = \frac{(x+1)^{\frac{3}{2}}(x-1)^{\frac{3}{4}}}{(x+2)^{\frac{1}{3}}}$ (b) $y = \sqrt{1-x}\,\sqrt{1-2x}\,\sqrt{1-3x}$
(c) $y = \frac{\sqrt{2x+1}}{(x-3)^2}$ (d) $y = \frac{\sqrt{2x+3}}{\sqrt{x^2-1}}$

276. (a) $y = \frac{(2x^2-1)^{\frac{1}{3}}}{(x-4)^{\frac{1}{5}}}$ (b) $y = (1+ax)(2+bx)$
(c) $y = \frac{(x^2+4)(x^3-1)}{(x+5)^2}$ (d) $y = \frac{x+1}{\sqrt{x+1}\,\sqrt{x+3}}$

277. (a) $y = \frac{\sqrt{x+13}}{(x-4)(2x+1)^{\frac{1}{3}}}$ (b) $y = \frac{x^2}{\sqrt{x^2+5x+1}}$
(c) $y = (x-1)\ln x$ (d) $y = x^2 \ln(x+1)$

278. (a) $y = \sqrt{\frac{1+x^2}{1-x^2}}$ (b) $y = \sqrt{x^3 - 2}\ \ln(x+19)$
(c) $y = \frac{x}{\ln \sqrt{x+1}}$ (d) $y = x^2 \ln \sqrt{3x+5}$

279. (a) $y = \frac{2x^2+1}{\ln(2x^2+1)}$ (b) $y = \frac{\ln(x+1)}{\ln(x-1)}$
(c) $y = \frac{(\ln(x-1))(\ln x)}{\ln(x+1)}$ (d) $y = [\ln(x-1)][\ln x^2] \ln \sqrt{x_1}$

280. (a) $y = \frac{\ln x}{x^2\sqrt{1+x^3}}$ (b) $y = \frac{\ln(x+a)}{x+a}$
(c) $y = \left(\frac{x^3}{\tan x}\right)^{\frac{1}{5}}$ (d) $y = \frac{(\sin x)^{\frac{2}{3}}}{(\cos x)^{\frac{5}{4}}}$

3.11 Tangent Lines

In problems 281 to 287 find the equation of the tangent line to the specified point on the function and give the coordinates of this point.

281. (a) $y = x^3$; at $x = 1$ (b) $y = 2x^2$; at $x = -2$
(c) $y = 4x + 5$; at $x = 4$ (d) $y = -x + e$; at $x = 1$

282. (a) $y = 3x - 2x^3$; at $x = 3$ (b) $y = x^3 = x^3 - x + 1$; at $x = -1$
(c) $y = x^2 - 2x$; at $x = 1$ (d) $y = \frac{x+1}{x-1}$; at $x = 2$

283. (a) $y = \sqrt{2x}$; at $x = 2$ (b) $y = \sqrt{3-x}$; at $x = -1$
(c) $y = \frac{1}{\sqrt{x-2}}$; at $x = 6$ (d) $y = x|x|$; at $x = -2$

284. (a) $y = x^2 \sin \frac{1}{x}$; at $x = \frac{2}{\pi}$ (b) $y = \frac{\sin 2x}{1+\cos 2x}$; at $x = \frac{\pi}{2}$
(c) $y = \frac{\sin 2x}{x}$; at $x = \frac{\pi}{2}$ (d) $y = \ln(\sin(ax))$; at $x = \frac{\pi}{4a}$

285. (a) $y = a(x - \sin x)$; at $x = 0$ (b) $y = \ln(\sin^2 x)$; at $x = \frac{\pi}{4}$
(c) $y = \sqrt{\cos 2x}$; at $x = \frac{\pi}{6}$ (d) $y = \ln(\sec x + \tan x)$; at $x = 0$

286. (a) $y = e^{2x}$; at $x = 0$ (b) $y = 2^{x+2}$; at $x = -2$
(c) $y = e^{\sqrt{2x}}$; at $x = 2$ (d) $y = x^a e^{bx}$; at $x = \frac{1}{b}$

287. (a) $y = e^x(x^2 + x + 1)$; at $x = 1$ (b) $y = \frac{e^x + e^{-x}}{2}$; at $x = 0$
(c) $y = \frac{e^x - e^{-x}}{2}$; at $x = 0$ (d) $y = 10^{x^2+1}$; at $x = -1$

288. (a) $y = \ln(x+1)$; at $x = 0$ (b) $y = \ln\left(\frac{x}{x-1}\right)$; at $x = 2$
(c) $y = x \ln x$; at $x = e^2$ (d) $y = \frac{\ln x}{x}$; at $x = 1$

In the following questions determine at what point or points **on** the curve does the tangent line have slope m. Your answer must include both coordinates of the point or points in question.

289. (a) Problem 281a, where $m = 0, -1$, and 2

    (b) Problem 281c, where $m = 0, 1$, and 4

    (c) Problem 282a, where $m = 0, 4$, and 1

    (d) Problem 282d, where $m = 1, -1, 0$, and -8.

290. (a) Problem 283a, where $m = 0, 1, -1$, and 2

    (b) Problem 283b, where $m = -\frac{1}{2}, \frac{1}{2}$, and $\frac{1}{4}$

    (c) Problem 283d, where $m = -1, 0, 1$, and 2.

291. (a) $y = \sin x$; where $m = -1, -\frac{1}{2}, 0, \frac{1}{2}, 1$, and 2

    (b) $y = 2\cos x$; where $m = -2, -1, -\frac{1}{2}, 0, 1$, and 2

    (c) $y = -4\sin x$; where $m = -4, -1, 0, \frac{1}{2}, 1, 2$, and 4

    (d) $y = \tan x$; where $m = \frac{1}{2}, 1, 10$, and 100.

292. (a) Problem 285d, where $m = 1, 2, -1$, and -2

    (b) Problem 286a, where $m = \frac{1}{2}, 1, 4$, and 10

    (c) Problem 287b, where $m = -1, 0, 1$, and 4

    (d) Problem 287d, where $m = 0$.

293. (a) Problem 288a, where $m = -1, 0, 1$, and 2

    (b) Problem 288b, where $m = -1, -\frac{1}{2}$, and 1

    (c) $y^3 = x^2$, where $m = -1$.

294. The tangent line to the graph f at the point P passes through the origin. Find the coordinates of P if

    (a) $f(x) = e^x$ (b) $f(x) = e^{-x}$

    (c) $f(x) = \ln x$ (d) $f(x) = \ln(x + 1)$

295. Find an equation for the line (or lines) tangent to the curve $y = x^3$ which passes through the point $(0, 2)$.

296. Show that the tangent to the curve $y = \ln x$ at any point can be constructed by joining the desired point of contact to the point on the y-axis which is a unit distance below the point of contact.

297. Show that the sum of the intercepts of any tangent to the curve $x^{\frac{1}{2}} + y^{\frac{1}{2}} = a^{\frac{1}{2}}$ is a constant.