Contents

Chapter 5

Rolle's Theorem, the Mean Value Theorem, and L'Hôpital's Rule

5.1 Rolle's Theorem

In the following problems

(a) Verify that the three conditions of Rolle's theorem have been met.

(b) Find all values z that satisfy the conclusion of the theorem.

1. (a) $f(x) = x^2 - 7x + 10,$ on $[2, 5]$ (b) $f(x) = x^2 - 7x + 10,$ on $[0, 7]$
   (c) $f(x) = x^2 - 7x + 10,$ on $[-1, 8]$

2. (a) $f(x) = x^2 - 4x,$ on $[0, 4]$ (b) $f(x) = x^2 - 4x,$ on $[-1, 5]$
   (c) $f(x) = x^2 - 4x,$ on $[-4, 8]$

3. (a) $f(x) = x^3 - 5x^2 - 17x + 21,$ on $-3 \le x \le 7$ (b) $f(x) = x^3 - 16x,$ on $-4 \le x \le 4$
   (c) $f(x) = x^3 + 2x^2 - x - 2,$ on $-2 \le x \le 1$

4. (a) $f(x) = x^2 - 6x - 7,$ on $[-1, 7]$ (b) $f(x) = x^3 + x^2 - 6x,$ on $[0, 2]$
   (c) $f(x) = x^2 - x - 2,$ on $[-1, 2]$ (d) $g(x) = x^3 + 5x^2 + 6,$ on $[-3, 0]$

5. (a) $f(x) = \sin 2x,$ on $[0, \pi]$ (b) $f(x) = \cos\left(\frac{x}{2}\right),$ on $[\pi, 3\pi]$
   (c) $f(x) = \sin x + \cos x$ on $\left[-\frac{\pi}{4}, \frac{3\pi}{4}\right]$
   $\left(\text{Hint: } \frac{3\pi}{4} = \frac{\pi}{4} + \frac{\pi}{2}\right)$

In problems 6 to 11 determine in what way the function fails to meet the conditions of Rolle's theorem.

6. $f(x) = 4x^2 - 8x \quad \text{for} \quad 0 \leq x \leq 3.$
7. $f(x) = 1 - \cos 4x + 2\cos x \quad \text{for} \quad \frac{3\pi}{2} \leq x \leq \frac{7\pi}{2}.$ $\left[f\left(\frac{3\pi}{2}\right) \neq 0\right]$
8. $f(x) = \frac{3x^2-2x+4}{x-2} \quad \text{for} \quad 1 \leq x \leq 5.$
9. $f(x) = \sin 2x + \cos 2x \quad \text{for} \quad 0 \leq x \leq \frac{3\pi}{8}.$
10. $f(x) = \frac{3x+2}{(x-1)^2} \quad \text{for} \quad \frac{1}{2} \leq x \leq 2.$
11. $f(x) = 1 - x^{\frac{2}{3}} \quad \text{for} \quad -1 \leq x \leq 1.$ $[f'(0)$ is undefined$]$
12. Find a function $f(x)$ on the interval $[-1, 1]$ such that $f(x)$ fails to meet the conditions of Rolle's theorem, but $[f(x)]^2$ does meet them. In what way does $f(x)$ fails to meet the conditions?
13. Use Rolle's theorem to prove that, regardless of the value of b, there is at most one point x in the interval $-1 \leq x \leq 1$ for which $x^3 - 3x + b = 0$.

5.2 Mean Value Theorem

For problems numbered 14 to 22,

(a) Verify that the conditions of the mean value theorem have been met.

(b) Find all numbers z that satisfy the theorem.

Example: $f(x) = x^3 - 2x^2 \quad \text{for} \quad 1 \leq x \leq 3.$

Solution.

(a) $f(x)$ is continuous (all polynomials are continuous)
$f'(x) = 3x^2 - 4x$ exists for $x \in (1, 3)$.
Thus the conditions of the mean value theorem have been met.

(b) $f(3) = 27 - 18 = 9.$
$f(1) = 1 - 2 = -1.$

Therefore,

$$\frac{f(b) - f(a)}{b-a} = \frac{f(3) - f(1)}{3-1} = \frac{9 - (-1)}{3-1} = \frac{10}{2} = 5$$

$$f'(z) = \frac{f(b) - f(a)}{b-a} \qquad \text{(M.V.T.)}$$

$$3z^2 - 4z = 5$$
$$3z^2 - 4z - 5 = 0$$

$$z = \frac{4 \pm \sqrt{16 - 4(-5)(3)}}{6}$$
$$= \frac{4 \pm \sqrt{76}}{6} \approx 2.1 \text{ or } -0.78$$

Since -0.78 is not in the interval discard it, but 2.1 is in the interval so that $z = \frac{4+\sqrt{76}}{6}$ is the answer.

14. $h(x) = \frac{x}{x+1}$ for $-\frac{1}{2} \le x \le \frac{1}{2}$
15. $f(x) = 3x + 4$ for $1 \le x \le 5$
16. $f(x) = x^3 - 3$ for $-3 \le x \le 1$
17. $f(x) = x^3$ for $-1 \le x \le 1$ $\left[z = \frac{1}{3}\sqrt{3}\right]$
18. $f(x) = x^3 - 2x + 1$ for $0 \le x \le 2$
19. $g(x) = x^{12}$ for $-23 \le x \le +23$
20. $r(x) = \frac{x}{x-1}$ for $2 \le x \le 3$
21. $f(x) = x^2 - 5x + 6$ for $1 \le x \le 5$ $[z = 3]$
22. $F(x) = x^3 - 4x^2 + 4x$ for $0 \le x \le 3$

For problems numbered 23 to 32 determine in what way each function fails to meet the conditions of the mean value theorem on the indicated interval.

23. $f(x) = \frac{x^2-2x}{x}$ for $-1 \le x \le 2$
24. $f(x) = \sqrt{4 - x^2}$ for $-2 \le x \le 5$
25. $f(x) = \frac{2}{\sin x}$ for $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$ [not continuous at $x = 0$]
26. $g(x) = 3x + 2$ for $-1 < x < 2$
27. $h(x) = \frac{(2x+3)}{(3x-2)}$ for $0 \le x \le 4$
28. $F(x) = \frac{(x-1)}{(x+1)}$ for $0 < x \le 3$
29. $f(x) = |x|$ for $-1 \le x \le 1$ [$f'(0)$ does not exist]
30. $f(x) = x^{\frac{2}{3}}$ for $-1 \le x \le 1$
31. $g(x) = |x - 1|$ for $-3 \le x \le 2$
32. $\phi(t) = \tan t$ for $0 \le t \le \pi$

For the problems numbered 33 to 55,

(a) Decide whether the mean value theorem applies to the function on the given interval.

(b) If it applies find z; if not, state why.

33. $H(w) = w^2 + 3w - 3$ for $0 \le w \le 4$ $[z = 2]$
34. $f(x) = x^2$ for $0 \le x \le 1$
35. $f(x) = -5x$ for $-1 \le x \le 1$
36. $y = \tan\theta$ for $\frac{-\pi}{4} \le \theta \le \frac{\pi}{4}$
37. $y = \sin(2\alpha)$ for $0 \le \alpha \le \pi$ $\left[z = \frac{\pi}{4} \text{ or } z = \frac{3\pi}{4}\right]$
38. $P(x) = \sqrt{x}$ for $-1 \le x \le 4$

39. $P(x) = \frac{x-1}{x+1}$ for $-2 \le x \le 2$
40. $f(x) = (2x+3)/(3x-2)$ for $1 \le x \le 4$
41. $h(x) = x^{\frac{2}{3}}$ for $-2 \le x \le 2$ [$h'(0)$ does not exist]
42. $f(x) = x^{\frac{2}{3}}$ for $0 \le x \le 2$
43. $g(x) = |x+2| - 1$ for $-2 \le x \le 2$
44. $S(x) = |x+2| - 1$ for $-3 \le x \le 2$
45. $r(x) = \frac{(x+1)}{(x-1)}$ for $2 \le x \le 3$ [$z = 1 + \sqrt{2}$]
46. $H(w) = 2w - \frac{1}{w}$ for $-1 \le w \le 1$
47. $f(x) = \sqrt{x-1}$ for $1 \le x \le 3$
48. $\phi(t) = \frac{(t+3)}{(t-3)}$ for $-1 \le t \le 4$
49. $F(x) = |x-2|$ for $0 \le x \le 4$ [$F'(2)$ doe snot exist]
50. $F(x) = |x-2|$ for $0 \le x \le 2$
51. $f(x) = x^3 - 2x^2 + x$ for $1 \le x \le 3$
52. $H(y) = \frac{2y-3}{y^2-1}$ for $1 \le y \le 4$
53. $B(x) = \begin{cases} x, & x \in [0,1) \\ 2, & x = 1. \end{cases}$ [discontinuity at $x = 1$]

    Interval for consideration is $0 \le x \le 1$.
54. $f(x) = \begin{cases} \dfrac{3x - x^2}{2}, & x \in [0,1) \\ \dfrac{1}{x}, & x \in [1,2] \end{cases}$. Interval for consideration is $0 \le x \le 1$.
55. $g(x) = 4x + 5$ for $1 \le x \le 3$
56. Show that on the graph of any quadratic polynomial the chord joining the points for which $x = a$ and $x = b$ is parallel to the tangent line at the midpoint $x = (a+b)/2$.
57. Suppose you know that F is continuous on $[a,b]$ and that it has a derivative on all of (a,b) for which $F'(x) = 0$ for every x. Show that F is a constant. (*Hint:* Take any $x \in (a,b]$. Show $F(x) = F(a)$.)
58. Suppose that in problem #57, we had said for every $x \in (a,b)$, $F'(x) = c$ where c is a nonzero constant. Show that the graph F is a straight line passing through $\big(a, F(a)\big)$. (*Hint:* Show that $F(x) = F(a) + c(x-a)$.)
59. Under the hypothesis of the Mean-Value Theorem with $I = [a, a+\Delta x]$ $(\Delta x \ne 0)$ prove there exists a number λ, $0 < \lambda < 1$, such that $f(a+\Delta x) - f(a) = \Delta x \cdot f'(a + \lambda \Delta x)$.
60. Determine λ, define in 59, in terms of "a" and Δx for $f(x) = x^2$ and $f(x) = x^3$. Find the limit of λ in each case as $\Delta x \to 0$.
61. A function f, continuous on $[a,b]$, has f'' everywhere in (a,b). The line joining the points $\big(a, f(a)\big)$ and $\big(b, f(b)\big)$ intersects the graph of f at a third point $\big(c, f(c)\big)$, where $a < c < b$. Prove $f'(t) = 0$ for at least one point t in (a,b).
62. Use theorems from this section to prove that an equation of the form $ax^2 + bx + c = 0$, where a, b, c are real numbers, can have no more than two real solutions.

63. (a) Suppose that f is a differentiable function on an interval and suppose $f'(x)$ is always between -1 and $+1$. Show that if x and y are in the interval, then

$$|f(x) - f(y)| \leq |x - y|.$$

(b) Show that

$$|\sin(x) - \sin(y)| \leq |x - y|$$

for all real numbers x and y.

5.3 L'Hospital's Rule

The correct spelling is either L'Hospital or L'Hôpital.

64. List the 3 conditions which must be met in order that L'Hospital's rule applies.

Compute the following limits and show your steps.

65. $\lim_{x\to 0} \frac{\tan x}{x}$

66. $\lim_{x\to 2} \frac{x^2-4}{x-2}$

67. $\lim_{x\to 0} \frac{\sqrt{1+t}-\sqrt{1-t}}{t}$

68. $\lim_{x\to 0} \frac{\tan x - x}{x - \sin x}$

69. $\lim_{x\to 0} \frac{\sin(5x)}{\sin(3x)}$

70. $\lim_{x\to 0} \frac{1-\cos x}{x^2}$

71. $\lim_{t\to 0} \frac{t\sin t}{1-\cos t}$

72. $\lim_{x\to 4} \frac{x^3-8x^2+2x+1}{x^4-x^2+2x-3}$

73. $\lim_{x\to 0} \frac{(\sin x)-2x}{x}$

74. $\lim_{x\to 0} \frac{\tan x}{4x}$

75. $\lim_{x\to 1} \frac{x^3-x}{x^4-2x^2+1}$

76. $\lim_{x\to 2} \frac{x^3-6x+4}{x^2+x-6}$

77. $\lim_{x\to 0} \frac{\sin 2x+\tan x}{3x}$

78. $\lim_{x\to 0} \frac{(\sin 2x)(\tan x)}{3x}$

79. $\lim_{x\to 0} \frac{\sin 2x+\tan x}{3x^2}$

80. $\lim_{u\to 0} \frac{\tan 2u}{u\sec u}$

From here on the problems will involve the indeterminate forms $\frac{0}{0}$, $\frac{\infty}{\infty}$, $\infty - \infty$, $0\cdot\infty$. *L'Hospital's rule works for only the indeterminate forms* $\frac{0}{0}$ *or* $\frac{\infty}{\infty}$.

Any questions that are of the $\infty - \infty$ form may be changed to a fraction by making a common denominator.

Any questions that are of the $0 \cdot \infty$ type may be rearranged by inverting part of it.

e.g. $\lim_{x\to 0} x \cdot \cot x = \lim_{x\to 0} \dfrac{x}{\left(\frac{1}{\cot x}\right)} = \lim_{x\to 0} \dfrac{x}{\tan x}$.

Now L'Hospital's rule may be applied.

81. $\lim_{x\to \frac{\pi}{2}} (\tan x - \sec x)$

82. $\lim_{x\to -3} \frac{x^2-9}{x+3}$

83. $\lim\limits_{x\to+\infty} \frac{x-2x^2}{x^2}$

84. $\lim\limits_{x\to-2} \frac{2x^2+5x+2}{x^2-4}$

85. $\lim\limits_{x\to1} \frac{x^3-3x+2}{x^3-x^2-x+1}$

86. $\lim\limits_{x\to\infty} \frac{2x^3-x^2+3x+1}{3x^3+2x^2-x-1}$

87. $\lim\limits_{x\to\infty} \frac{x^3-3x+1}{2x^4-x^2+2}$

88. $\lim\limits_{x\to0} \frac{\tan 3x}{\sin x}$

89. $\lim\limits_{x\to a} \frac{x^p-a^p}{x^q-a^q},\ a>0$

90. $\lim\limits_{x\to\frac{\pi}{2}} \frac{\sec^2 3x}{\sec^2 x}$

91. $\lim\limits_{x\to-\infty} \frac{x^4-2x^2-1}{2x^3-3x^2+3}$

92. $\lim\limits_{x\to0} \frac{\sin 7x}{x}$

93. $\lim\limits_{x\to\frac{\pi}{2}} \frac{1-\sin x}{\cos x}$

94. $\lim\limits_{x\to\frac{\pi}{2}} \frac{\cos x}{\sin^2 x}$

95. $\lim\limits_{x\to\infty} \frac{\sin x}{x}$

96. $\lim\limits_{x\to\frac{\pi}{2}} \frac{\sin x}{x}$

97. $\lim\limits_{x\to0} x\cot x$

98. $\lim\limits_{x\to\frac{\pi}{2}} \left(x-\frac{\pi}{2}\right)\sec x$

99. $\lim\limits_{\theta\to0} \left(\csc\theta-\frac{1}{\theta}\right)$

100. $\lim\limits_{x\to0} \left(\cot^2 x-\frac{1}{x^2}\right)$

101. $\lim\limits_{x\to\infty} \frac{3x^2+x}{5x^2-1}$

102. $\lim\limits_{x\to-\infty} \frac{2x^2+4}{3x^2-5x+2}$

103. $\lim\limits_{x\to1^+} \frac{x^4-3x^3+3x^2-3x+2}{x^6-2x^5+x^4-7x^3+14x^2-7x}$

104. $\lim\limits_{x\to-1} \frac{x^4-3x^2-2x}{x^5+2x^4+x^3-x^2-2x-1}$

105. $\lim\limits_{x\to\frac{\pi}{2}} \frac{\cos^2 x}{x-\left(\frac{\pi}{2}\right)}$

106. $\lim\limits_{x\to0^+} \frac{1}{x^2}-\frac{1}{\sin x}$

107. $\lim\limits_{x\to0} \frac{(\tan x)-x}{x^2}$

108. $\lim\limits_{x\to0} \frac{\cos^2 x-1}{\sin x}$

109. $\lim\limits_{x\to0^-} \frac{\sin x-2\cos 2x}{x^3\cos x}$

110. $\lim\limits_{x\to\frac{\pi}{4}} \frac{\sec x-\sqrt{2}}{\tan x-1}$

111. $\lim\limits_{x\to\frac{\pi}{3}} \frac{2\cos x-1}{2\sin x-\sqrt{3}}$

112. $\lim\limits_{x\to0} \frac{\sin^2 x-x^2\cos^2 x}{x^2\sin^2 x}$

113. $\lim\limits_{x\to0} \frac{x\csc x-1}{x^2}$

114. $\lim\limits_{x\to0^+} \left(\tan x-\frac{1}{x}\right)$

115. $\lim\limits_{x\to0}(\sin x)\left(\frac{\csc x-1}{x^3}\right)$

116. $\lim\limits_{x\to\frac{\pi}{2}^-} (\sec^3 x-\tan^3 x)$

117. $\lim\limits_{x\to0} x^2(\csc^2 x-\csc x\cot x)$

118. $\lim\limits_{x\to a^-} (1-2x)\tan\pi x$ where $a=\frac{1}{2}$

119. $\lim\limits_{x\to0} \frac{\sin x-x+(x^3/6)}{x^5}$

120. $\lim\limits_{t\to1} \frac{nt^{n+1}-(n+1)t^n+1}{(t-1)^2}$

121. $\lim\limits_{x\to\infty} \frac{1+\cos 2x}{1-\sin x}$

122. $\lim\limits_{x\to\frac{\pi}{2}} \frac{\sec x+1}{\tan x}$

123. $\lim\limits_{x\to\frac{\pi}{2}} \frac{\cos x}{x-\frac{\pi}{2}}$

124. $\lim\limits_{x\to\frac{\pi}{2}} (\sec x-\tan x)$

125. L'Hospital's Rule is *not* all-powerful!
Show that the limit $\lim_{x\to\infty} \frac{2x-\sin x}{x+\sin x}$ exists but cannot be evaluated using L'Hospital's Rule.

126. $\lim_{x\to\infty} \frac{\ln x^{100}}{x}$

127. $\lim_{x\to+\infty} \left(1+\frac{1}{x}\right)^x$

128. $\lim_{x\to 1} \frac{\ln x}{\sqrt{1-x}}$

129. $\lim_{x\to+\infty} \frac{\ln x}{x}$

130. $\lim_{x\to 0} \frac{\ln x}{x}$

131. $\lim_{x\to+\infty} \frac{\ln x}{x^n}$ (n a positive integer)

132. $\lim_{x\to 0} \frac{a^x-b^x}{x}$

133. $\lim_{x\to 0^+} x\ln x$

134. $\lim_{x\to 0} \frac{e^x-e^{-x}}{4\sin x}$

135. $\lim_{x\to\infty} \frac{x\ln x}{x^2+1}$

136. $\lim_{x\to 0} \frac{\ln\cos 3x}{2x^2}$

137. $\lim_{t\to 1} \frac{\sqrt{t}-t}{\ln t}$

138. $\lim_{x\to 0} \frac{\sin^2 3x}{x(1-e^x)}$

139. $\lim_{x\to 0} 3x\ln(1-e^{2x})$

140. $\lim_{x\to 0} \frac{\ln\tan x}{\ln\tan 2x}$

141. $\lim_{x\to 0^+} \frac{1-\ln x}{e^{\frac{1}{x}}}$

142. $\lim_{x\to 1} \frac{x^x-x}{1-x+\ln x}$

143. $\lim_{x\to 0^+} \frac{e^{-\frac{1}{x}}}{x}$ (Hint: use the substitution $u=\frac{1}{x}$)

144. $\lim_{x\to\infty} \frac{2x}{\ln(3x+e^x)}$

145. $\lim_{x\to\infty} \frac{x^3}{e^x+x^2}$