Contents

Chapter 7

Estimating Functions

7.1 Estimating Functions by Simpler Functions

Preliminaries

You will also need the following results and the material given in your lectures.

A. Let n be a positive integer.
If $|x| \leq 1$, then $|x| \geq |x|^n$.
If $|x| \geq 1$, then $|x| \leq |x|^n$.
If $|x| \leq 1$, then $|x| \leq |x|^{\frac{1}{n}}$.
If $|x| \geq 1$, then $|x| \geq |x|^{\frac{1}{n}}$.
If $|x| \leq |y|$ and $a > 0$ then $|x|^a \leq |y|^a$.

B. If $|x| \geq 1$, use $|\sin x| \leq 1$ and $|\cos x| \leq 1$.
If $|x| \leq 1$, use $|\sin x| \leq |x|$.

C. $|x| \leq \sqrt{1+x^2} \leq |x| + 1$.

Problems

1. Prove that for any real numbers x and y

$$|x| - |y| \leq |x + y|.$$

2. Using that the square root function is an increasing function show that

$$\sqrt{|x|} \leq \sqrt{1+|x|} \leq 1 + \frac{|x|}{2}.$$

Hint: compute $\left(1 + \frac{|x|}{2}\right)^2$.

3. Show that if $|x| \leq 1$ then $2|x| \leq |x^4 + 3x| \leq 4|x|$.

4. Show that if $|x| \leq 1$ then $1-6|x| \leq |x^4+3x^3-2x+1| \leq 6|x|+1$. Conclude that $0 \leq |x^4+3x^3-2x+1| \leq 7$ on the interval $[-1, 1]$.

5. Show that if $|x| \le 1$ then $0 \le |x^5 - x^4 + x^3 + x - 1| \le 4|x| + 1$.
6. Show that if $|x| < \frac{1}{4}$ then $1 - 4|x| \le |x^5 - x^4 + x^3 + x - 1| \le 4|x| + 1$.
7. Show that if $|x| \le 1$ then $0 \le |3x^3 - 2x^2 - x + \sin x| \le 7|x|$.
8. Show that if $|x| \le 1$ then $2|x| \le |3x^3 - 2x^2 + 8x + \sin x| \le 14|x|$.
9. Show that if $|x| \le \frac{1}{10}$ then $0 < 5 - 10|x| \le |2x^2 - 8x + 5| \le 10|x| + 5$.
10. Show that if $|x| \le 1$ then $0 \le |\sin x + \cos x| \le |x| + 1$.
11. Show that if $|x| \le 1$ then $0 \le |5\sin x + 3\cos x| \le 5|x| + 3$.
12. Show that if $|x| \le 1$ then $1 - x^2 \le |x| + 1$.
13. Show that $\sqrt{x^2 + 2x + 2} \le |x| + 2$.
14. Show $e^{\frac{x}{2}} \le \sqrt{(\sin^2 x) + e^x} \le 1 + e^{\frac{x}{2}}$. Conclude that in $[0,1]$ $1 \le \sqrt{(\sin x)^2 + e^x} \le 1 + \sqrt{e}$.
15. Show that $\sqrt{\sin^2 x + 2\cos^2 x} \le \sqrt{2}$.
16. On the interval $[-1,1]$ show that $\sqrt{1 - x^2} \le \cos x \le 1$. Graph the three functions on a common coordinate system.
17. On the interval $[-1,1]$ show that $1 - x^2 \le \sqrt{1 - x^2} \le 1$. Graph the three functions on a common coordinate system.
18. On the interval $[-1,1]$ show that $1 - |x| \le 1 - x^2$. Graph both functions together on a common coordinate system.
19. If $|f_1(x)| \le 2$ on $[0,2]$ and $|f_2(x)| \le x$ on $[1,3]$ what can you say about $|f_1 f_2|$ on $[1,2]$?
20. Show $|x^2 + x + 2| \le 2|x|^2$ if $|x| \ge 2$. What can you say about $|(x^2 + x + 2)\sin \pi x|$ on $[2, \infty)$?
21. (a) On the interval $(-1,1)$, show that $\frac{1}{(1-x^2)(1+x^2)} \ge 1$.

    (b) On the interval $(-1,1)$ show that $\frac{1}{1+x^2} \ge 1 - x^2$.
22. On the interval $(-1,1)$ show that $\frac{\cos x}{\sqrt{1-x^2}} \ge 1$.
23. If $x \in (-1,1)$, show that $\frac{\cos x}{1+x^2} \ge (1 - x^2)\cos x$.
24. If $x \in (-1,1)$, show that $\frac{x}{1-x^2} \ge x - x^3$.
25. If $x \in (-a,a)$, show that $\frac{1}{a+|x|} \le \frac{1}{a+x} \le \frac{1}{a-|x|}$.
26. Show that $|\cos z| + |\sin z| \le \sqrt{2}$. (*Hint:* Square both sides.)

Next we wish to estimate the given function, $|P(x)|$, from above and below over the interval $|x| \ge a$ by functions of the form $c|x|^n$. In the formula $c|x|^n$, c is a constant and n is the highest power of x occurring in $P(x)$.

In problems 27 to 38 compute c_0 and c_1 such that $c_0|x|^n \leq |P(x)| \leq c_1|x|^n$ for each of the given intervals $|x| \geq a$.

Note: c_0 and c_1 are not unique.

27. $P(x) = x^3 + 7x^2 - 3$. $a = 10$.

    Solution: If $|x| \geq 10$, $|x|^3 - |7x^2 - 3| \leq |x^3 + 7x^2 - 3|$

    $|x|^3 - \{7|x|^2 + 3\} \leq |x^3 + 7x^2 - 3|$

    (We used the inequality $7|x|^2 + 3 \geq |7x^2 - 3|$.)

    Now (i) $7|x|^2 = \frac{7|x|^2|x|}{|x|} \leq \frac{7|x|^3}{10}$ since $|x| \geq 10$,

    and (ii) $3 = \frac{3|x|^3}{|x|^3} \leq \frac{3|x|^3}{1000}$.

    Therefore $|x|^3 - \frac{7|x|^3}{10} - \frac{3|x|^3}{1000} \leq |x^3 + 7x^2 - 3|$

    $\frac{297}{1000}|x|^3 \leq |x^3 + 7x^2 - 3|$.

    Hence $\underline{c_0 = \frac{297}{1000}}$ if $|x| \geq 10$.

    Observe that the solution consists of an ordered sequence of steps leading to a stated conclusion.

    Next we estimate $|x^3 + 7x^2 - 3|$ from above:

    $|x^3 + 7x^2 - 3| \leq |x|^3 + 7|x|^2 + 3$ by the triangle inequality $|x^3 + 7x^2 - 3| \leq |x|^3 + \frac{7}{10}|x|^3 + \frac{3|x|^3}{1000}$

    (same type of argument as in the first part of the solution.)

    $|x^3 + 7x^2 - 3| \leq \frac{1703}{1000}|x|^3$

    Hence $c_1 = \frac{1703}{1000}$ if $|x| \geq 10$.

28. $P(x) = x^3 + 7x^2 - 3$; $a = 2$, and $a = 50$.
29. $P(x) = x^2 - 8x$; $a = 2$, $a = 10$, and $a = 100$.
30. $P(x) = x^2 + 17x - 4$; $a = 3$, $a = 27$, and $a = 1024$.
31. $P(x) = x^3 - \sin x$; $a = 1$, $a = 16$, and $a = 512$.
32. $P(x) = x^4 + 3x \sin x$; $a = 1$, $a = 2$, and $a = 1000$.
33. $P(x) = x^3 - \sin x + \cos x$; $a = 2$, $a = 2^5$, and $a = 2^{10}$.
34. $P(x) = x^2 + \sqrt{1 + x^2}$; $a = 1$, $a = 10$, and $a = 10^4$.
35. Graph $|\log x|$ and $I(x) = x$ on a common coordinate system. Observe the fact that $|\log x| \leq x$ if $x \geq 1$.
36. $P(x) = x^2 + \log x$; $a = 1$, $a = 2$, and $a = 150$. (See question #35.)
37. $P(x) = x^3 + \frac{\log x}{x}$; $a = 1$, $a = 16$, and $a = 512$. (See question #35.)
38. $P(x) = x + x \log x$; $a = 1$, $a = 8$, and $a = 1024$. (See question #35.)

In problems 39 to 45 a function f is defined and two constants, c_0 and c_1 are given. Write a solution to find a real number b such that if $|x| \geq b$ then $c_0|x|^n \leq |f(x)| \leq c_1|x|^n$.

39. $f(x) = ax^3 + 2x + 4$; $c_0 = |a| - .4$ and $c_1 = |a| + .5$.

Solution: Part I

Step 1 If $|x| \geq 1$ then

$$|ax^3 + 2x + 4| \leq |a|\,|x|^3 + 2|x| + 4$$

$$|ax^3 + 2x + 4| \leq |a|\,|x|^3 + 2|x| + 4|x|$$

$$|ax^3 + 2x + 4| \leq |a|\,|x|^3 + 6|x| = \left(|a| + \tfrac{6}{|x|^2}\right)|x|^3$$

Step 2 If $|x| \geq b_1 \geq 1$ then

$$|a| + \tfrac{6}{|x|^2} \leq |a| + \tfrac{6}{b_1^2} \text{ and so}$$

$$|ax^3 + 2x + 4| \leq \left(|a| + \tfrac{6}{b_1^2}\right)|x|^3.$$

Step 3 We want $|ax^3 + 2x + 4| \leq (|a| + .5)|x|^3$. This condition will be true if $|x| \geq b_1 \geq 1$ and b_1 is a number such that $\left(|a| + \frac{6}{b_1^2}\right) \leq |a| + .5$.

Now $|a| + \frac{6}{b_1^2} \leq |a| + .5 \Rightarrow \frac{6}{b_1^2} \leq .5 \Rightarrow b_1^2 \geq 12$. If we choose $b_1 = \sqrt{12}$ then indeed $\left(|a| + \frac{6}{b_1^2}\right) \leq |a| + .5$. Thus if $|x| \geq \sqrt{12}$ then $|ax^3 + 2x + 4| \leq (|a| + .5)|x|^3$.

Part II

Step 1 If $|x| \geq 1$ then

$$|a|\,|x|^3 - |2x + 4| \leq |ax^3 + 2x + 4|$$

$$|a|\,|x|^3 - 6|x| \leq |ax^3 + 2x + 4|$$

$$\left(|a| - \tfrac{6}{|x|^2}\right)|x|^3 \leq |ax^3 + 2x + 4|.$$

Step 2 If $|x| \geq b_2 \geq 1$ then

$$|a| - \tfrac{6}{|x|^2} \geq |a| - \tfrac{6}{b_2^2} \text{ and so}$$

$$\left(|a| - \tfrac{6}{b_2^2}\right)|x|^3 \leq |ax^3 + 2x + 4|.$$

Step 3 We want $(|a| - .4)|x|^3 \leq |ax^3 + 2x + 4|$. This condition will be true if $|x| \geq b_2 \geq 1$ and b_2 is a number such that $|a| - \frac{6}{b_2^2} \leq |a| - .4$.

Now $|a| - \frac{6}{b_2^2} \leq |a| - .4 \Longrightarrow \frac{6}{b_2^2} \geq .4 \Longrightarrow b_2^2 \leq 15$. Thus if $b_2 \leq \sqrt{15} \Longrightarrow |a| - .4 \leq |ax^3 + 2x + 4|$.

Part III

Since the condition $b_1 \geq \sqrt{12}$ and $b_2 \leq \sqrt{15}$ defines a non empty interval $\left[\sqrt{12}, \sqrt{15}\right]$ a solution exists. Let $b = \min\left\{\sqrt{12}, \sqrt{15}\right\} = \sqrt{12}$. If $|x| \geq \sqrt{12}$ then $|ax^3 + 2x + 4| \leq (|a| + .5)|x|^3$ by Part I and $|a| - .4 \leq |ax^3 + 2x + 4|$ by Part II.

The method used to solve problems 27 and 39 does not work for all estimating problems. Even if the method does work a different solution is acceptable if it is supported by a logical argument.

For example consider this problem: Find a number b, if one exists, such that whenever $|x| \geq b$ then

$$|x|^3 \leq |x^3 + 2x + 4| \leq 1.01|x|^3.$$

Solution:

<u>Part I</u>: $|x|^3 \leq |x^3 + 2x + 4|$.

If $c_0 < 1$ we could use the techniques illustrated in the solution to #39. When $c_0 = 1$ this is not possible. We will use the definition of absolute value to get started.

Case 1: $x > 0 \Rightarrow x^3 + 2x + 4 > 0$. So $x^3 < x^3 + 2x + 4 \Rightarrow |x|^3 \leq |x^3 + 2x + 4|$.

Case 2: If $x^3 + 2x + 4 < 0$ then $|x|^3 \leq |x^3 + 2x + 4| \Rightarrow -x^3 \leq -x^3 - 2x - 4$. The solution set for this inequality is $x \leq -2$. We must check that our condition $x^3 + 2x + 4 < 0$ holds for $x \leq -2$. $x^3 + 2x + 4 < 0$ is true for $x < 0$ and $2x + 4 < 0$. $2x + 4 \leq 0 \Rightarrow x \leq -2$. Therefore our condition is valid for $x \leq -2$. Observe we did not need to solve the inequality $x^3 + 2x + 4 < 0$. Both cases hold for $|x| \geq 2$. Therefore the function is bounded below for $b = 2$.

<u>Part II</u>: $|x^3 + 2x + 4| \leq 1.01|x|^3$.

If $|x| > 1$, then since

$$\begin{aligned} |x^3 + 2x + 4| &\leq |x|^3 + 2|x| + 4 \\ &\leq \left(1 + \frac{2}{|x|^2} + \frac{4}{|x|^3}\right)|x|^3 \end{aligned}$$

We need only choose b so that $1 + \frac{2}{|x|^2} + \frac{4}{|x|^3} \leq 1.01$. To see this, observe that for this choice of b

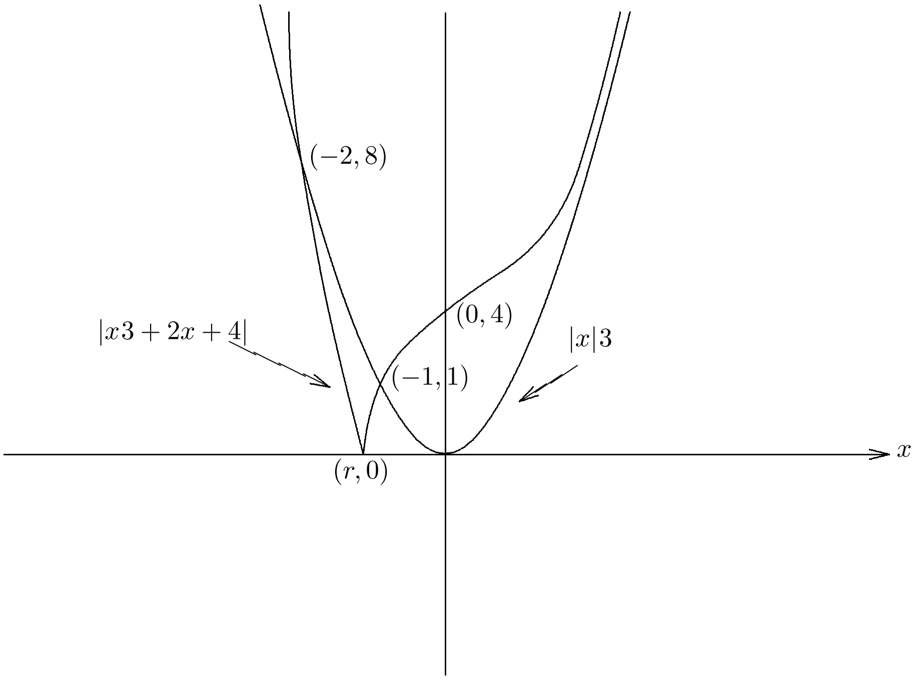
$$|x^3 + 2x + 4| \leq \left(1 + \frac{2}{|x|^2} + \frac{4}{|x|^3}\right)|x|^3 \leq 1.01|x|^3.$$

$b = 100$ will work (a smaller b would do: try $b = 15$).

Conclusion:

Since $|x| \geq 2$ is needed for part I to work, $|x| > 100$ will work for part I as well. Choose $b = 100$.

I have included a graph of $|x|^3$ and $|x^3+2x+4|$ to illustrate part I. Let $f(x) = x^3+2x+4$. $f(-1) = 1$ and $f(-2) = -8$. Let r be the root of f; i.e., $f(r) = 0$. Therefore $-2 < r < -1$.



40. $f(x) = ax^3+2x+4$; $c_0 = |a| - .5$ and $c_1 = |a| + .5$
41. $f(x) = ax^2+2x-1$; $c_0 = |a| - .005$ and $c_1 = |a| + .005$
42. $f(x) = ax^2+2x-1$; $c_0 = |a| - .5$ and $c_1 = |a| + 2$
43. $f(x) = ax^3+x^2-x+\sin x$; $c_0 = |a| - .01$ and $c_1 = |a| + .01$
44. $f(x) = ax^3+x\sqrt{1+x^2}$; $c_0 = |a| - .5$ and $c_1 = |a| + .55$
45. $f(x) = ax^3 - x\sin x$; $c_0 = |a|$ and $c_1 = |a| + .01$

In problems 46 to 56 see if you can find a number b such that $c_0|x|^n \le |f(x)| \le c_1|x|^n$ when $|x| \ge b$. If you cannot try to explain why.

46. $f(x) = 5x^3+2x+4$; $c_0 = 3$ and $c_1 = 4$
47. $f(x) = 5x^3+2x+4$; $c_0 = 4.4$ and $c_1 = 5.5$
48. $f(x) = -5x^3+2x+4$; $c_0 = 4.5$ and $c_1 = 5.5$
49. $f(x) = x^3+2x+4$; $c_0 = 1$ and $c_1 = 1.01$ (see above solution)
50. $f(x) = x^3+2x+4$; $c_0 = .99$ and $c_1 = 1.01$
51. $f(x) = -x^3+2x+4$; $c_0 = .99$ and $c_1 = 1$
52. $f(x) = \frac{x^3}{2}+1$; $c_0 = .99$ and $c_1 = 1$
53. $f(x) = 4x^3-x+1$; $c_0 = 3.5$ and $c_1 = 4.5$

54. $f(x) = 2x + 3x \sin x$; $c_0 = 1$ and $c_1 = 1.5$

55. $f(x) = x + 3x \sin x$; $c_0 = 2.9$ and $c_1 = 3.1$

56. $f(x) = x^3 + \sqrt{x^2 + 1}$; $c_0 = .5$ and $c_1 = 1.5$

57. Write a solution to find a real number b such that if $|x| \geq b$ then

$$|x| - 0.1 \leq \left|\frac{1}{x} + x\right| \leq |x| + .001$$

58. Write a solution to find a real number b such that if $|x| \geq b$ then

$$|x|^2 - .01 \leq \left|x^2 + \frac{1}{x^2+1}\right| \leq |x|^2 + .01$$

59. Write a solution to find a real number b such that if $|x| \geq b$ then

$$|x| - .001 \leq \left|\frac{x^2 + x + 3}{x + 1}\right| \leq |x| + .001$$

60. Write a solution to find a real number b such that

$$0 \leq \left|\sin x + \frac{1}{x}\right| \leq 1.01$$

61. $f(x) = x^3 + x^2 + 4$; $c_0 = 1$ and $c_1 = 1.1$

7.2 A Strategy for Computing Bounds on $|f(x)|$

Find an upper bound for $|g(t)|$ on the interval $[0, \frac{1}{2}]$ where $g(t) = \frac{24 - 240t^2 + 120t^4}{(1+t^2)^5}$.

Solution:

$$\begin{aligned} |g(t)| &= \frac{1}{|(1+t^2)^5|}|24 - 240t^2 + 120t^4| \\ |g(t)| &= \frac{24}{(1+t^2)^5}|1 - 10t^2 + 5t^4| \quad (1 + t^2 > 0 \text{ for all } t.) \end{aligned}$$

<u>Method</u>

Given a ratio $\frac{f}{h}$ of two function f and h. Compute an upper bound M for $|f|$ and a lower bound N for $|h|$ over the given interval. Then $\left|\frac{f}{h}\right| \leq \frac{M}{N}$ on the interval.

Write $f(t) = 24 - 240t^2 + 120t^4$ and $h(t) = (1 + t^2)^5$.

Bound $|f|$:

$$\begin{aligned} |f| &= 24|1 - 10t^2 + 5t^4| \\ |f| &\leq 24(1 + |-10t^2| + |5t^4|) \\ |f| &\leq 24(1 + 10t^2 + 5t^4). \end{aligned}$$

Since $1 + 10t^2 + 5t^4$ is an increasing function on $[0, \frac{1}{2}]$, $f(\frac{1}{2}) = 24\left(\frac{61}{16}\right) = \frac{183}{2}$. So $|f| \leq \frac{183}{2}$. Pick $M = \frac{183}{2}$.

Bound $|h|$:

Let $|h| = (1+t^2)^5$. Since $1+t^2 > 0$ for all $t \in \mathbb{R}$. h is an increasing function. Reason: $|h|' = 10t(1+t^2)^4 > 0$ because $t > 0$ on $[0, \frac{1}{2}]$. Therefore a lower bound for $|h|$ on $[0, \frac{1}{2}]$ is $h(0) = 1$.

Conclusion:

(a) $|g(t)| = \left|\frac{f}{h}\right| \leq \frac{\left(\frac{183}{2}\right)}{1} = 91.5.$

(b) A *sharper bound* may be obtained as follows: Let $f(t) = 24(1 - 10t^2 + 5t^4)$. $f'(t) = 24(-20t + 20t^3) = 484t(t^2 - 1)$. $f'(t) < 0$ when $t^2 - 1 < 0$ and $t > 0$. This occurs for $t \in (0, 1)$. Thus on $[0, \frac{1}{2}]$ f is decreasing. Now $f(0) = 24$ and $f(\frac{1}{2}) = -\frac{57}{2}$. Therefore the maximum value of $|f|$ on $[0, \frac{1}{2}]$ is $\frac{57}{2} < 29$. Pick $M = 29$. For this choice of M $|g(t) = \left|\frac{f}{h}\right| < \frac{29}{1} = 29$.

Note: If f is decreasing on an interval, the maximum value of f occurs at one of the interval's endpoints.

<u>Example 1</u>: How to find an upper bound for $|g(x)| = \frac{|x+1|}{|x+2|^2}$ on $[-1, 1]$. Maximize the numerator and minimize the denominator. Since $x + 2 \geq 0$ for $x > -2$, $|x + 2| = x + 2$ on $[-1, 1]$. $x + 2$ is an increasing function. (It has positive slope = 1.) Therefore $|x + 2|^2$ has a minimum value of 1. Since $x + 1 \geq 0$ for $x \geq -1$, $|x+1| = x+1$ on $[-1, 1]$. This function is also increasing with maximum value of 2. Thus $|g(x)| \leq 2$ on $[-1, 1]$.

Note $|g(-1)| = 0$ and $|g(1)| = \frac{2}{9}$. But $|g(x)|$ maximum value does not occur at an endpoint but at $x = 0$, with value of $\frac{1}{4} > \frac{2}{9}$! To show this we note that

$$\begin{aligned} |g(x)| &= \frac{x+1}{(x+2)^2} \qquad \text{on } [-1, 1] \\ |g(x)|' &= \frac{(x+2)^2 - (x+1)(x+2)}{(x+2)^4} = \frac{-x}{(x+2)^3} \end{aligned}$$

It is easy to show that $|g(x)|' > 0$ on $(-2, 0)$. (You could use the chart method for example) and $|g(x)|' < 0$ on $(0, \infty)$. Therefore $|g(x)|$ is increasing on $(-2, 0)$ and decreasing on $(0, \infty)$ so the maximum occurs at $x = 0$, NOT AT ANY ENDPOINT.

It is generally much easier to get an upper bound, by maximizing the numerator and minimizing the denominator because the derivative of $|g(x)|'$ may be much more complicated than in this example.

<u>Example 2</u>: Find M such that $f(x) = \frac{|x \cos x + \sin x|}{|x \sin x|} < M$ on $I = [3.75, 4.75]$.

<u>Numerator</u>: $|z \cos z + \sin z| = |\sqrt{z^2+1}\ \sin(z + \alpha)| = \sqrt{z^2+1}\ |\sin(z + \alpha)|$.

(Recall that $a \cos\theta + b \sin\theta = \sqrt{a^2+b^2}\ \sin(\theta + \alpha)$ where $\sin\alpha = \frac{a}{\sqrt{a^2+b^2}}$ and $\cos\alpha = \frac{b}{\sqrt{a^2+b^2}}$). But $|\sin\theta| \leq 1$ for all $\theta \in \mathbb{R}$. Therefore $|z \cos z + \sin z| \leq \sqrt{z^2+1}$. Since both the square root function and z^2+1 are increasing functions on I, $|z \cos z + \sin z| \leq \sqrt{(4.75)^2 + 1} < 4.86$.

<u>Denominator</u>: $|x \sin x| = |x||\sin x| \geq 3.73|\sin x|$.

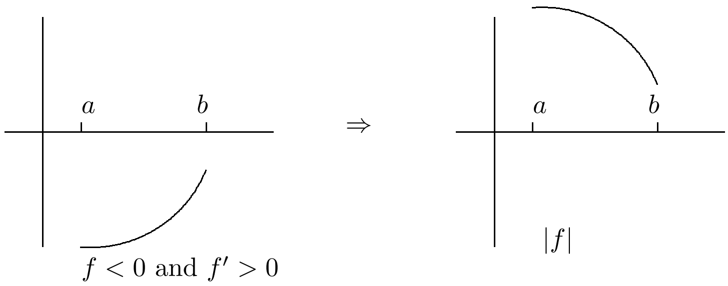
Since $\pi < 3.73 < 4.74 < 2\pi$ the minimum of $|\sin x| = \min\{|\sin 3.75|, |\sin 4.75|\} = \min\{0.571, 0.999\} > 0.57$. (To see this look at graph of $|\sin x|$ on $[\pi, 2\pi]$.)

Therefore the denominator is greater than 3.75×0.57. Thus

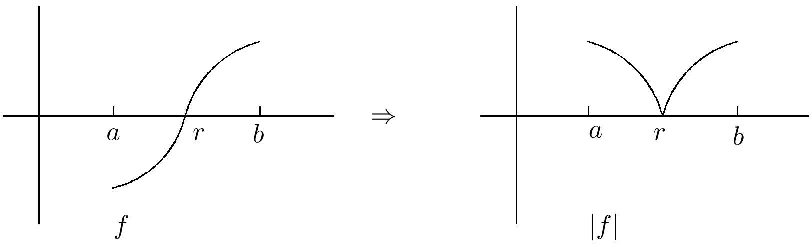
$f(x) \leq \frac{5}{(3.75)(0.57)} < 2.27$. We can choose $M = 2.27$.

$|f|' = \frac{f}{|f|} f'$.

If $f < 0$ on $[a, b]$ and $f' > 0$ or $f' < 0$ on $[a, b]$ but not both. Then (1) if $f' > 0$, $|f|$ is decreasing on $[a, b]$ or (2) if $f' < 0$ $|f|$ is increasing on $[a, b]$.



If $f(a)f(b) < 0$ and $f' > 0$ or $f' < 0$ on $[a, b]$ then the maximum of $|f|$ will occur at an endpoint and the minimum is zero.



Problems

In questions 62 to 71 compute the smallest upper bound for $|f|$ on the given interval. (Smallest means the best you can do with the techniques illustrated in the sample solutions.)

62. $f(x) = \frac{1}{(x-1)^3} - \frac{21}{(x+1)^2}$ on $[-\frac{1}{2}, \frac{1}{2}]$
63. $f(x) = \frac{x^3-6x+1}{\sin x}$ on $[\frac{1}{2}, 1]$
64. $f(x) = \frac{\sin x + 2\cos x - x^2}{(x-1)^2}$ on $[3, 4]$
65. $f(x) = \frac{1}{(x^2+1)^2} + \frac{7}{\cos 2x + 1}$ on $[0, \frac{\pi}{2}]$
66. $f(x) = \frac{x \ln x}{x + \ln x}$ on $[\frac{1}{2}, 1]$
67. $f(x) = e^{-x}(x^2 - x + 1)$ on $[-5, -4]$
68. $f(x) = \frac{6 \sin x}{3x^2+2}$ on $[-1, 1]$
69. $f(x) = \frac{1}{x^{\frac{7}{4}}} - \frac{1}{x^{\frac{3}{2}}}$ on $[1, \frac{3}{2}]$
70. $f(x) = \frac{\sqrt{1+x^2}}{\ln x}$ on $[2, e]$

71. $f(x) = \frac{x\sqrt{x}}{x^2-10x+1}$ $\left[\frac{1}{10}, \frac{1}{2}\right]$

$$\underline{|f''(z)| < B \quad \text{on} \quad [a,b]}$$

In Problems 72 to 85 compute the smallest upper bound for $|f''(z)|$ on the given interval. (Smallest means the best you can do with the techniques illustrated in the sample solutions and your lectures.)

Example: For the function $g(x) = \sqrt{1 + \left(\frac{1}{1+x^2}\right)^2}$ compute B such that $|g''(z)| \leq B$ on the interval $0 \leq z \leq 1$.

Solution:

Write $f(z) = 1 + \frac{1}{(1+z^2)^2}$. Then $g(z) = \sqrt{f(z)}$.

Compute $g''(z)$:

$$\begin{aligned}
g'(z) &= \frac{f'(z)}{2\sqrt{f(z)}} \\
g''(z) &= \frac{2\sqrt{f(z)}\, f''(z) - f'(z)\left(\frac{f'(z)}{\sqrt{f(z)}}\right)}{4f(z)} \\
g''(z) &= \frac{f''(z)}{2\sqrt{f(z)}} - \frac{(f'(z))^2}{4(f(z))^{\frac{3}{2}}} \\
|g''(z)| &\leq \left|\frac{f''(z)}{2\sqrt{f(z)}}\right| + \left|\frac{(f'(z))^2}{4(f(z))^{\frac{3}{2}}}\right|
\end{aligned}$$

Computer f' and f'':

$$\begin{aligned}
f'(z) &= -\frac{4z}{(1+z^2)^3} \\
f''(z) &= \frac{-4(1+z^2)^3 + (4z)(6z)(1+z^2)^2}{(1+z^2)^6} \\
f''(z) &= \frac{4(1+z^2)^2(6z^2 - (1+z^2))}{(1+z^2)^6} \\
f''(z) &= \frac{4(5z^2-1)}{(1+z^2)^4}
\end{aligned}$$

Bound $\left|\frac{f''(z)}{2\sqrt{f(z)}}\right|$ *and* $\left|\frac{(f'(z))^2}{4(f(z))^{\frac{3}{2}}}\right|$

First

$$\left|\frac{f''(z)}{2\sqrt{f(z)}}\right| = \frac{2|5z^2-1|}{(1+z^2)^4\sqrt{f(z)}} \leq \frac{2(5z^2+1)}{(1+z^2)^4\sqrt{f(z)}}.$$

Since $f(z)$ is decreasing on $[0,1]$ and $f(z) > 0$ on $[0,1]$, the smallest value $\sqrt{f(z)}$ attains on $[0,1]$ is $\sqrt{f(1)} = \frac{\sqrt{5}}{2}$.

Also $(1+z^2)^4$ minimum value is 1. Therefore

$$2\left[\frac{5z^2+1}{(1+z^2)^4\sqrt{f(z)}}\right] < \frac{2(5z^2+1)}{\sqrt{\frac{5}{2}}} < 10.73.$$

Next

$$\left|\frac{(f'(z))^2}{4(f(z))^{\frac{3}{2}}}\right| = \frac{16z^2}{4(1+z^2)^6(f(z))^{\frac{3}{2}}} < \frac{4z^2}{(f(z))^{\frac{3}{2}}}.$$

As before $f(z)$ is decreasing $\Rightarrow (f(1))^{\frac{3}{2}} = \left(\frac{\sqrt{5}}{2}\right)^3$ is the smallest value for the denominator. Hence

$$\left|\frac{(f(z))^2}{4(f(z))^{\frac{3}{2}}}\right| < \frac{4}{\left(\frac{\sqrt{5}}{2}\right)^3} < 2.862.$$

Therefore $|g''(z)| \le 10.8 + 2.9 = 13.7$.

An alternative solution is:

Write $g''(z) = \frac{2f(z)f''(z)-(f'(z))^2}{4(f(z))^{\frac{3}{2}}}$. Then compute the numerator:

$$\begin{aligned} 2f(z)f''(z) - (f'(z))^2 &= \frac{2((1+z^2)^2+1)}{(1+z^2)^2}\left(\frac{4(5z^2-1)}{(1+z^2)^4}\right) - \frac{16z^2}{(1+z^2)^6} \\ &= \frac{8(((1+z^2)^2+1)(5z^2-1)-2z^2)}{(1+z^2)^6} \\ &= \frac{8(5z^6+9z^4+6z^2-2)}{(1+z^2)^6}. \end{aligned}$$

Therefore

$$\begin{aligned} g''(z) &= \frac{2(5z^6+9z^4+6z^2-2)}{(1+z^2)^6(f(z))^{\frac{3}{2}}} \\ |g''(z)| &\le \frac{2|5z^6|+|9z^4|+|6z^2|+|-2|}{(1+z^2)^6(f(z))^{\frac{3}{2}}}. \end{aligned}$$

An upper bound for the numerator on $[0,1]$ is $2(5+9+6+2) = 44$. A lower bound for the denominator may be obtained by replacing $(1+z^2)^6$ and $(f(z))^{\frac{3}{2}}$ with their minimum values on $[0,1]$. They are 1 and $(f(z))^{\frac{3}{2}} = \left(\frac{\sqrt{5}}{3}\right)^3$ since f is decreasing function on $[0,1]$.

Thus $|g''(z)| \le \frac{44}{\left(\frac{\sqrt{5}}{2}\right)^3} < 31.49$. But if you simplify the denominator as follows:

$$(1+z^2)^6\left(\frac{(1+z^2)^2+1}{(1+z^2)^2}\right)^{\frac{3}{2}} = (1+z^2)^3((1+z^2)^2+1)^{\frac{3}{2}}$$

you find the minimum of this product is $2^{\frac{2}{3}}$ and $|g''(z)| \le \frac{44}{2^{\frac{2}{3}}} < \underline{15.6}$.

72. $f(x) = \sin x^2$ on $[0,\pi]$
73. $f(x) = K\ln(100^2 + (500-44x)^2)$ on $[0,10]$
74. $f(x) = \frac{1}{\sqrt{1-z^2}}$ on $[-\frac{1}{2},\frac{1}{2}]$
75. $f(x) = \sqrt{1+x^3}$ on $[0,\frac{1}{2}]$
76. $f(x) = \pi\sqrt{16-3x^2}$ on $[0,2]$

77. $f(x) = \sqrt{1 + \left(\frac{1}{1+x}\right)^4}$ on $[9, 39]$
78. $f(x) = \frac{1}{\sqrt{1+z+z^3}}$ on $[1, 2]$
79. $f(x) = \arctan \sqrt{x}$ on $[9, 13]$
80. $f(x) = e^{\sqrt{x}}$ on $[1, 4]$
81. $f(x) = \cos x^2$ on $[0, \sqrt{\frac{\pi}{2}}]$
82. $f(x) = \sqrt{1 + \cos^2 x}$ on $[0, \frac{\pi}{2}]$
83. $f(x) = \sqrt{\sin x}$ on $[\frac{\pi}{4}, \frac{\pi}{2}]$
84. $f(x) = e^{-x^2}$ on $[0, 1]$
85. $f(x) = \frac{\sin x}{x}$ on $[\frac{\pi}{2}, \pi]$