Contents

8 Applications of Maxima and Minima **127**

8.1 The Use of Auxiliary Variables 137

Chapter 8

Applications of Maxima and Minima

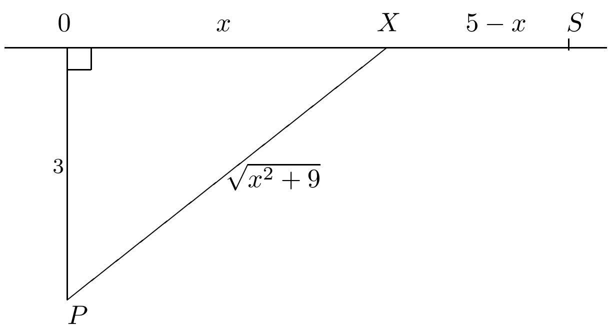
Prerequisites

When you write a solution, it is your responsibility to inform the reader what you are doing. You may assume the marker is intelligent but not a mind reader. Therefore you must invest some time in the organization and documentation of your work. Study the following problems and solution with these points in mind.

PROBLEM

A lighthouse is at point P, 3 miles offshore, from the nearest point 0 of a straight beach. A store is located 5 miles down the beach from 0. The lighthouse keeper can row 3.25 miles/hour and walk 4 miles/hour. How far along the beach from 0 should he land in order to get to the store in the minimum time?

Solution:



Let x be the distance from 0 to the point of landing X and let S be the location of the store 5 miles from 0.

A. Compute the time function for the trip:

The time required to row from P to X is $\frac{\sqrt{x^2+9}}{3.25}$. (Recall that velocity = distance/time). The time required to walk from X to S is $\frac{5-x}{4}$. The total time, T, for the trip is given by:

$$T(x) = \frac{\sqrt{x^2+9}}{3.25} + \frac{5-x}{4}.$$

In this problem it only makes sense for x to vary from 0 to 5 miles. Therefore we need to find the minimum of T on the closed interval $[0, 5]$.

B. Compute the critical points of the function:

$$\frac{dT}{dx} = \frac{x}{3.25\sqrt{x^2+9}} - \frac{1}{4}.$$

Therefore

$$\begin{aligned} \frac{dT}{dx} &= 0 \Leftrightarrow 4x = 3.25\sqrt{x^2+9} \\ &\Leftrightarrow 16x^2 = (3.25)^2(x^2+9) \\ &\Leftrightarrow \{16-(3.25)^2\}x^2 = 9(3.25)^2 \\ &\Leftrightarrow x^2 \cong 17.482759 \\ &\Leftrightarrow x \cong 4.1812389 \quad \text{or} \quad -4.1812389 \end{aligned}$$

C. Evaluate the function at the critical and endpoints to find the value of x that minimizes the trip.

$$\begin{aligned} T(0) &\cong 2.1730769 \\ T(5) &\cong 1.794139 \\ T(4.1812389) &\cong 1.788118 \end{aligned}$$

D. Conclusion.

Choose $x = 4.1812389$ to minimize the time of the trip. Observe that a very slight increase in rowing speed would make rowing directly to S the fastest route.

The following problems may be done by using a four step technique which is outlined below.

Step 1

(a) Draw a diagram. The diagram should be *large* and *neatly* drawn, occupying about $\frac{1}{3}$ page.

(b) *Label* fixed quantities with numerical values. Label the dimensions that vary with letters. Choose letters for any other items that are involved in the problem, e.g. volume.

Step 2

(a) Select the quantity that is to be made maximum or minimum. Express it as a function of other quantities.

(b) If you are to maximize something which is a function of *two* variables, then you must find a relationship between the two variables. Usually the geometry of the situation gives a relation. (e.g. similar triangles, Pythagoras theorem.)

(c) Express the quantity to be maximized as a function of only *one* variable.

Step 3

Differentiate with respect to the variable. Find all values where the derivative is equal to zero and any values where the derivative is undefined but the function is defined. These values are called the critical points of function.

Step 4

(a) Use either the first derivative test or the second derivative test to determine whether critical values are relative maxima or relative minima.

(b) Determine the domain of the function and note any restrictions on this domain imposed by the problem. Evaluate the function at the critical points (if they lie in the domain) and the endpoints (if they exist) to determine the maximum or minimum.

(c) If there are no endpoints; that is, the domain is not a closed interval a minimum or maximum may still exist. So you must give an argument different from part b to support your conclusion.

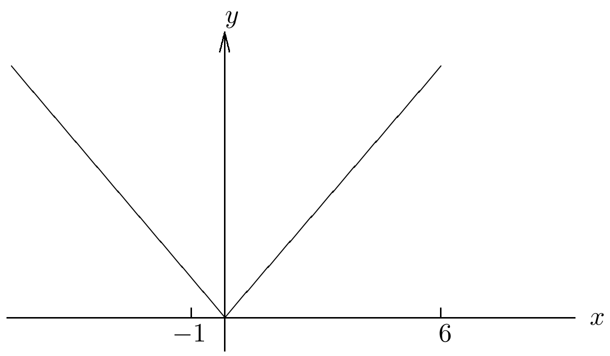
(d) Read the question again to see what quantity was asked for and finish the question that was asked. (For example, it is not correct to give a critical "radius" as the answer if the maximum "volume" was asked.)

Examples

1. Find the maximum and minimum value of $A(x) = |2x|$ on the interval $[-1, 6]$.

   Solution:

   Since $A(x) = \begin{cases} 2x, & \text{if } x \geq 0 \\ -2x, & \text{if } x < 0. \end{cases}$ by definition of absolute value, the graph of $A(x)$ is



   $A'(x) = \begin{cases} 2, & \text{if } x > 0 \\ -2, & \text{if } x < 0. \end{cases}$ **NOTE:** $A'(x)$ is undefined at $x = 0$ but $A(0) = 0$. Therefore $x = 0$ is a critical point. the minimum occurs at this point, $x = 0$ and is equal to zero. $A(-1) = 2$ and $A(6) = 12$. Therefore the maximum occurs at the endpoint, $x = 6$ and is equal to 12.

2. Find the maximum and minimum value of $A(x) = |2x|$ on the interval $(-\infty, \infty)$.

   Solution:

   $A'(x) = \begin{cases} 2, & \text{if } x > 0 \\ -2, & \text{if } x < 0. \end{cases}$ and so the only critical value is $x = 0$ as seen in example 1. On $(-\infty, 0)$, $A'(x) = -2$ implies A is a continuous and decreasing function for $x < 0$. On $(0, \infty)$, $A'(x) = 2$ implies A is continuous and increasing for $x > 0$. Therefore $A(0) = 0$ is the minimum and A has no maximum value.

3. Find the minimum of $f(x) = x^2 - 2x + 1$.

   Solution:

   The domain is $(-\infty, \infty)$.

   $f(x) = x^2 - 2x + 1 = (x-1)^2$.

   But $(x-1)^2 \geq 0$ for all x and $f(1) = (1-1)^2 = 0$.

   Hence the minimum value of $f(x)$ is 0 on $(-\infty, \infty)$.

4. Show that the maximum value of f on the interval $(-\infty, \infty)$ is 1 where $f(x) = e^{-x^2}$.

   Solution:

   $f'(x) = -\frac{2x}{e^{x^2}}$. So we can conclude the following:

   (a) $f'(x) = 0 \Rightarrow x = 0$.

   (b) Since $e^{x^2} > 0$ for all x, $f'(x)$ exists for all x.

   (c) Part b implies $x = 0$ is the only critical point and f is continuous on $(-\infty, \infty)$.

   (d) Since $-\frac{2x}{e^{x^2}} > 0 \Rightarrow x < 0$, f is strictly increasing on $(-\infty, 0)$.

   (e) Since $-\frac{2x}{e^{x^2}} < 0 \Rightarrow x > 0$, f is strictly decreasing on $(0, \infty)$.

   (a) d and e imply $f(0) = 1$ is the only maximum for f on $(-\infty, \infty)$.

1. A man has a stone wall alongside a field. He has 1200 ft. of fencing material and he wishes to make a rectangular pen, using the wall as one side. What should the dimensions of the pen be in order to enclose the largest possible area?

   Step 1. Draw a diagram showing a rectangular pen alongside the stone wall. Label the ends x.

   Step 2. If the man has 1200 ft. of fence and each end requires x ft., how long may the other side be?

   Step 3. Differentiate the area function with respect to x. Let $D_xA = 0$. What are critical values?

   Step 4. Do critical values give a relative maximum? Use first derivative test.
   What is the domain of x? Would the end-points of the domain of x give a maximum value for the area? Finally answer the question "What should the dimensions of the pen be?"

2. A rectangle has a perimeter of 120 ft. What length and width yield the maximum area? What is the result when the perimeter is L units?

   Step 1. Draw a diagram of a rectangle. Label one side x.

   Step 2. If the perimeter is 120 ft., what is the length of the other side?
   What is the area of the rectangle as a function of x Finish it.

3. Find the dimensions of the rectangle of maximum area that can be inscribed as a circle of radius 6. What is the result for a circle of radius R?

   Step 1. Draw a diagram. You should have a co-ordinate axis with a circle about the origin and a rectangle in the circle (oriented so that it's symmetric about both axes.) Label the point where the top right hand corner of the rectangle touches the circle as (x, y).

   Step 2. What is the area of the rectangle as a function of x and y? What is the equation of a circle? Express y as a function of x. What is the area of the rectangle as a function of x only?

   Step 3. Differentiate the area function with respect to x. Let $D_xA = 0$. Find the critical values of x.

   Step 4. (a) Determine if the critical value gives a maximum. Note what is the domain of x. What values would the area have if x has the end-point values?

   (b) Write a short sentence saying what is the maximum area.

4. The sum of one number and three times a second number is 60. Among the possible numbers which satisfy this condition, find the pair whose product is as large as possible.

   Step 1. Let one number be x and the other number be y. Let P be the product.

   Step 2. What quantity is to be maximized? Express the product P as a function of x and y. Write down a relationship between x and y as given in the very first sentence of the question. Express P as a function of x only. Finish it.

5. Find the dimensions of the right circular cylinder of maximum volume which can be inscribed in a sphere of radius 12.

   Step 1. Draw a diagram of the side view of a cylinder insider a sphere. Label the radius of the base of the cylinder r and the height of the cylinder as h. These two dimensions are variables because the cylinder may be tall and thin or short and fat as it fits inside the sphere. The radius of the sphere is fixed at 12. Label it.

   Step 2. What is the volume V of the cylinder as a function of r and h? Using Pythagoras' Theorem relate r and h in an equation. What is the volume V of the cylinder as a function of r? Finish it.

6. An eavestrough is to be made from a long piece of sheet iron 8 in. wide by turning up equal widths along the edges into vertical position. How many inches should be turned up at each side to yield the maximum carrying capacity?

   Step 1. Draw a diagram of the cross-section of the eavestrough. The drawing should look like a rectangle, open at the top. Label the length sides turned up both x.

   Step 2. Finish it.

7. The sum of twice one number and five times a second number is to be 70. What numbers should be selected so that the product of the numbers is as large as possible?

   Step 1. Let one number be x and the other number be y. Let the product be P.

   Step 2. What quantity is to be maximized? Express the P as a function of x and y. Write down a relationship between x and y as given in the very first sentence of the question. Express P as a function of x only. Finish it.

8. The difference between two numbers is 20. Select the numbers so that the product is as small as possible. (You should be able to do this question easily if you have already done # 7.)

9. A box is to be made from a piece of sheet metal 12 in. square by cutting equal small squares from each corner and turning up the edges. Find the dimensions of the box of largest volume which can be made in this way.

   Step 1. Draw the diagram. You should get a square 12 in. to a side with 4 little squares, one in each corner, cut out. Label the side of one little square as x. Draw the box you would get if you fold up the edges. (The box has no lid.) Label the length, width and the height of the box you would get.

   Step 2. What is the volume of the box as a function of x?

10. A rectangular box with an open top is to be made in the following way. A piece of tin 10 in. by 16 in. has a small square cut from each corner, and then the edges are folded vertically. What should be the size of the squares cut out if the box is to have as large a volume as possible?

    Step 1. Draw a diagram of a rectangle with a square in each corner which will be the part cut out. Let the side of each square be x. Draw a picture of the box you would get when you fold up the flaps. Label the length, width and height of the box.

Step 2. What is the volume of the box as a function of x? Finish it.

11. A box with a square base is to have an open top The area of the material in the box is to be 100 in^2. What should the dimensions be in order to make the volume as large as possible? What is the result for an area of S square inches?

    Step 1. Draw a box with a square base. Label the length and width of the box x. Label the height y.

    Step 2. What is the volume of the box as a function of x and y? What is the surface area of the box as a function of x and y? The surface area S is fixed, so you now may write y as a function of x. What is the volume of the box as a function of x only?

12. Find the two positive numbers whose sum is 30 having maximum product.

    Step 1. Let the two numbers be x and y. Let the product be P.

    Step 2. Express P as a function of x and y. Express P as a function of x only.
    Finish it.

13. A soup manufacturing company wishes to pack 25 cu. in. of mushroom soup in a can in the form of a right circular cylinder. Find the dimensions of the can if the surface area is to be a minimum.

    Step 1. Draw the cylindrical can. Label the radius r and the height h. These are both variable quantities. Call the volume V and surface area S.

    Step 2. What is the volume of the can as a function of r and h? What is the surface area of the can as a function of r and h? The volume V is fixed, so that you may write h as a function of r. What is the surface area of the can as a function of r only?

14. A box with square base is to be constructed to hold 64 cu. in. Find the dimensions of the box of minimum surface area.

    Step 1. Draw the box. Read carefully to see what is square. Label the sides of the square x. Label the other dimension y.

    Step 2. What is the surface area of the box as a function of x and y? Volume is fixed at 64 cu. in., so that you may write y as a function of x. What is the surface area of the box as a function of x only?

15. A sheet of paper for a poster is 18 sq. ft. in area. The margins at the top and bottom are 9 in. each and the margin on each side is 6 in. What are the dimensions of the paper if the printed area is maximum?

    Step 1. Draw the diagram. Label the fixed distances in either *feet* or *inches* (but be consistent in using your choice for the rest of the problem.) Label the length and width of the poster x and y. These are variables.

    Step 2. What is the area of the printed part as a function of x and y? Find a relationship between x and y using the first sentence of the question. Write x in terms of y. What is the area of the printed part as a function of x only?

16. Find the dimensions of the isosceles triangle of maximum area if the perimeter is to be 24 inches.

    Step 1. Draw an isosceles triangle. Label the base as x and one of the sides which is different in size from x as y. These sides are able to vary in size.

    Step 2. Express the perimeter as a function of x and y. Express the area as a function of x and y. Using the perimeter equation find y in terms of x. Then substitute into the area equation to get area as a function of x only. Finish it.

17. Find the dimensions of the rectangle of largest area that can be inscribed in a semicircle having diameter $2r$.

    Step 1. Draw a semicircle having a diameter of $2r$, on a co-ordinate axis with the origin at the centre of the circle. Draw the rectangle with one side along the diameter. Since it is a maximum, corners of the rectangle will touch the semi-circle. Label the point (x, y) where the rectangle touches the circle in the first quadrant.

    Step 2. What is the area of the rectangle as a function of x and y?

    What is the equation of a circle? Using this equation, find y as a function of x.

    What is the equation of the rectangle as a function of x? Finish it.

18. Find the dimensions of the rectangle of largest area that can be inscribed in the ellipse; $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

    Step 1. Draw an ellipse on co-ordinate axes centred on the origin. (The ellipse crosses the x-axis at $-a$ and $+a$, and it crosses the y-axis at $-b$ and $+b$.) Draw a rectangle inside the ellipse. Let it be oriented so that it is symmetric about both axes. The rectangle should touch the ellipse. Label the corner where it touches the ellipse in the first quadrant as (x, y).

    Step 2. Express the area of the rectangle as a function of both x and y.

    Using the equation of the ellipse given in the problem, find y in terms of x.

    Now express the area of the rectangle as a function of x only. Finish it.

19. Consider an isosceles triangle with sides 5, 5, and 6. Find the dimensions of the rectangle of largest area that can be inscribed in the triangle so that one side is along the base.

    Step 1. Draw a diagram with the base of the triangle being 6. Draw the rectangle inside the triangle with one side of the rectangle lying along the 6 inch side. Label the sides of the rectangle x and y, and label fixed dimensions of the triangle.

    Draw a line from the middle of the base to the point at the top of the triangle. How long is this line? Label it with its length.

    Step 2. What is the area of the rectangle as a function of x and y?

    Look for similar triangles. Using ratios of the sides, find a relationship between x and y. Write y as a function of x from this relation. Express the area of the rectangle as a function of x only. Finish it.

20. A piece of wire 20 in. long is to be cut in two pieces, one to form a circle and the other a square. How should the wire be cut in order that the sum of the two areas enclosed by the wire be minimal?

    Step 1. Draw a circle of radius r and square with side x.

    Step 2. Write an expression for the perimeter of the square plus the perimeter of the circle as a function of x and r. Write an expression for the area of the square plus the area of a circle as a function of x and r. Using the perimeter equation, express x as a function of r. Substitute this function into the area function to get area as a function of r only. Finish it.

21. A cylindrical can having 18 cu. in. volume is to be covered by a label on the side but not on the circular ends. What should the dimensions of the can be to minimize the surface of the label?

    Step 1. Draw a cylinder and its rectangular label. Call the radius of the cylinder r and the height of it x. Call the dimensions of the label x and y.

    Step 2. There are 3 variables x, y, and r. Express area as a function of x and y. Express the volume of the cylinder as a function of r and x. The y dimension of the label is the circumference of the can so write down this relationship between y and r. There are now 3 equations. Express the area as a function of one variable only. Finish it.

22. An open cylindrical tank with circular base is to be constructed of sheet metal so as to contain a volume πa^3 of water. Find the height and the radius of the base so that the quantity of sheet metal required may be minimal.

    Step 1. Draw a diagram of the cylindrical tank. Label the radius r and the height h.

    Step 2. Express the volume of the cylinder of water as a function of r and h. Express the surface area of the tank as a function of r and h. The volume πa^3 is fixed, so express r in terms of h. What is the surface area of the tank as a function of h only? Finish it.

23. A 288 cu. ft. pool is to have a square top. The sides are to be built of see-through glass and the bottom of mosaic. The cost per unit area of glass is three times the cost of mosaic. Find the dimensions of the pool having minimum cost.

    Step 1. Draw a diagram. Let the height of the pool be h and both the length and width of the base be x.

    Step 2. Find the volume function and the cost function. Finish it.

24. A Norman window is in the shape of a rectangle surmounted by a semicircle. Find the dimensions when the perimeter is 12 ft. and the area is as large as possible.

    Step 1. Draw the diagram. There are 3 straight sides from part of a rectangle and the top side is arched in a semicircle. Let the width be x and the length of the straight portion of the height be y.

    Step 2. Write the perimeter as a function of x and y. Write the area as a function of x and y. The perimeter has a fixed value. Using the perimeter function find an expression for y in terms of x. Write the area as a function of x only. Finish it.

25. At midnight, ship B was 90 mi. due south of ship A. Ship A sailed east at 15 mi/hr and ship B sailed north at 20 mi/hr. At what time were they closest together?

    Step 1. For the diagram use co-ordinate axes. Start ship A at the origin. At that same moment, what are the co-ordinate of ship B?

    After a while A has sailed to a point x mi. along the x axis. Label the co-ordinates of this point. B is at the same moment, at a point y (from the origin) along the y-axis. Label the co-ordinates of this point. Call the distance from A to B, S.

    Step 2. Find S^2 as a function of x and y.

    Step 3. Differentiate S^2 with respect to time t, and minimize S^2. (Notice that differentiating S^2 instead of S is mechanically easier.) Finish it.

26. One end of a cantilever beam of length L is built into a wall, while the other end is simply supported. If the beam weighs w lb. per unit length, its deflection y at distance x from the built-in end satisfies the equation

    $$48EIy = w(2x^4 - 5Lx^3 + 3L^2x^2),$$

    where E and I are constants that depend on the material of the beam and the shape of its cross section. How far from the built-in end does the maximum deflection occur?

27. Determine the constant a so that the function:

    $$f(x) = x^2 + \frac{a}{x}$$

    may have

    (a) a relative minimum at $x = 2$,

    (b) a relative minimum at $x = -3$,

27. (c) the second derivative is 0 at $x = 1$.

    Show that the function cannot have a relative maximum for any value of a.

28. Determine the constants a and b so that the function

$$f(x) = x^3 + ax^2 + bx = c$$

    may have

    (a) a relative maximum at $x = -1$ and a relative minimum at $x = 3$,

    (b) a relative minimum at $x = 4$ and the second derivative is 0 at $x = 1$.

29. The distance between the points (x_1, y_1) and (x_2, y_2) is

$$\left((x_2 - x_1)^2 + (y_2 - y_1)^2\right)^{\frac{1}{2}}.$$

    Find the point on the curve $y = x$ nearest the point $(c, 0)$

    (a) if $c = \frac{1}{2}$ (b) if $c = -\frac{1}{2}$.

30. A certain generator with an internal resistance of r ohms delivers E volts. This generator is connected to an electric circuit with R ohms resistance. The work W done each second in sending a current through a circuit with resistance R ohms is given by $W = [E^2R/(r + R)^2]10^7$ ergs. For constant r and E, show that W is a maximum when $R = r$.

31. Find the point of the graph of the equation

$$y = x^2$$

    that is nearest the point $A = (3, 0)$.

32. A ladder is to reach over a fence 8 ft. high to a wall 1 ft. behind the fence. What is the length of the shortest ladder that can be used?

33. A real estate office handles 80 apartment units. When the rent of each unit is \$180.00 per month, all units are occupied. However, for each \$6 increase in rent, one of the units becomes vacant. Each occupied unit requires an average of \$18 per month for service and repairs. What rent should be charged to realize the most profit?

34. Three sides of a trapezoid have the same length a. Of all such possible trapezoids, show that the one of maximum area has its fourth side of length $2a$.

35. A Boston lodge has asked the railroad company to run a special train to New York for its members. The railroad company agrees to run the train if at least 200 people will go. The fare is to be \$8 per person if 200 go, and will decrease by 1¢ for everybody for each person over 200 that goes (thus, if 250 people go, the fare will be \$7.50). What number of passengers will give the railroad maximum revenues?

36. Find the co-ordinates of the point or points on the curve $y = 2x^2$ which are closest to the point $(9, 0)$.

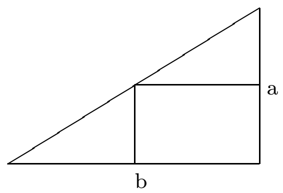
37. Find the co-ordinates of the point or points on the curve $x^2 - y^2 = 16$ which are nearest to the point $(0, 6)$.

38. Find the co-ordinates of the point or points on the curve $y^2 = x + 1$ which are nearest to the origin.

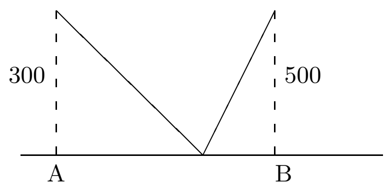
39. Find the co-ordinates of the point or points on the curve $y^2 = \frac{5}{2}(x + 1)$ which are nearest to the origin.

40. Two cars are travelling along two roads which cross each other at right angles at A. Both cars are travelling toward A at 30 ft. per sec. Initially their distances from A are 1500 ft. and 2100 ft. respectively. At what time is the distance between the two cars a minimum? Find this distance.

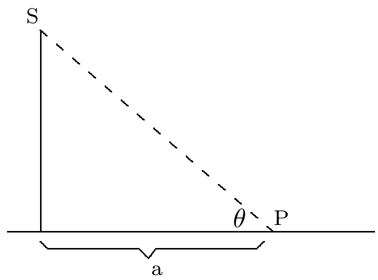
41. (a) A right triangle has a hypotenuse of length 13 and one leg of length 5. Find the dimensions of the rectangle of largest area which has one side along the hypotenuse and the ends of the oppositive side on the legs of triangle.
    (b) What is the result for a hypotenuse of length H with an altitude to it of length h?
42. A trough is to be made from a long strip of sheet metal 12 in. wide by turning up strips 4 in. wide on each side so that they make the same angle with the bottom of the trough (trapezoidal cross section). Find the width across the top such that the trough will have maximum carrying capacity.
43. The sum of three positive numbers is 30. The first plus twice the second plus three times the third add up to 60. Select the numbers so that the product of all three is as large as possible.
44. The sum of three positive numbers is 40. The first plus three times the second plus four times the third add up to 80. Select the numbers so that the product of all three is as large as possible.
45. A rectangular box with square bottom and top is to contain 1000 cu. ft. The cost of material per square foot for the bottom is 25¢, for the top, 15¢, and for the sides, 20¢. The labour charge for making the box is \$3. Find the dimensions of the box when the cost is minimal.
46. A rectangle is inscribed as shown, in a right triangle. Find the rectangle of maximum area.



47. Two houses are 300 yd. and 500 yd. from a straight power line and are 800 yd. apart measured along the power line. Where should they attach to the power line to make the total length of cable a minimum?



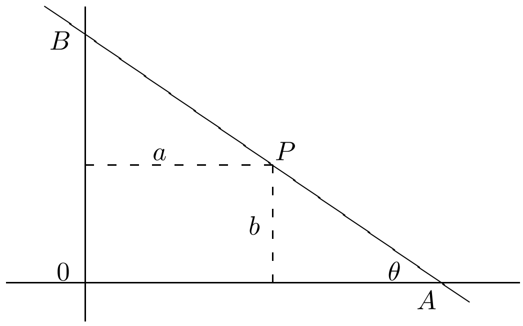
48. What point on the parabola $y^2 = 2px$ is nearest the point $(a, 0)$, where $a > p > 0$.
49. What is the least distance from a point on the line $ax \downarrow$ by $\uparrow$ $c = 0$ to the origin?
50. Radiation intensity on the ground at P is directly proportional to $\sin\theta$ and inversely proportional to the square of the distance from the source S. What height for the source gives maximum intensity at P?



51. Determine θ so that $|AB|$ is a minimum.

52. Determine θ so that $|OA| + |OB|$ is a minimum.

53. Determine θ so that $|OA| \cdot |OB|$ is a minimum.



54. Find the point on the x axis the sum of whose distances from the points $(2, 0)$ and $(0, 3)$ is a minimum.

55. A space capsule is in the form of a sphere of radius r. What is the volume of the largest astronaut which can be put inside the capsule? You may assume that an astronaut has the shape of a right circular cylinder.

56. A snail, who can travel at 3 feet per hour (running), is 10 miles from the nearest point P on a straight highway. She wishes to travel to a point Q 50 miles along the highway from P, and she can arrange to have a turtle pick her up anywhere along the highway. The turtle travels at 60 feet per hour. What point on the highway should the snail head for so as to arrive at Q in the shortest possible time?

57. The strength of a beam of rectangle cross section is proportional to its breadth times the cube of its depth. What are the dimensions of the strongest beam that can be cut from a log of radius r?

58. A man lives on a plain at point $(3, 4)$. There is a highway along the graph of $f(x) = x^3$. He wants to make the shortest lane possible between his house and the highway. Using calculus, tell him where the lane should meet the highway.

59. A cow has 90 ft. of fencing. She decides to make two pastures, one circular and one square. There are no openings. What is the maximum total pasture she can obtain?

8.1 The Use of Auxiliary Variables

At the beginning of the maximum minimum problems, 4 steps were suggested as a good method of approach. At the end of Step 2, you have up until now substituted into the function for the quantity to be maximized or minimized in such a way as to make the function involve *only one* variable. If this is very awkward or even impossible, the way around it is to differentiate implicitly. With this approach the new steps are as follows:

Step 2

Write down the various functions given by the wording of the problem, the geometry of the diagram, etc. There should be as many functions as there are variables.

By substituting one relation into another reduce the number of variables as far as possible.

Now the quantity to be maximized (call it A) is a function of more than one variable. One variable is singled out and all other variables are to be *thought* of as functions of this one variable.

Step 3

(a) Differentiate all of the relations in Step 2 *implicitly* with respect to the one variable which has been singled out.

(b) Eliminate by substituting in expressions for the derivatives which can be found from the extra relations into the expression for A'.

Step 4

Same as before.

Step 5

Let $A' = 0$ and solve for initial points. Consider the critical points and the end-points of the domain to find the maximum or minimum.

Answer the question that was asked in the first place.

Caution: In this type of solution 90% of the errors occur in Step 3. About 60% are from incorrectly differentiating implicitly. About 30% are from failing to eliminate the extra variables. This suggests that the students lack practice with the techniques rather than the intelligence to think through the problem!

Solve the following maximum-minimum problems using the implicit method.

60. Suppose that a closed right circular cylinder (i.e., top and bottom are included) has a surface area of 100 in^2. What should the radius and altitude be in order to provide the largest possible volume?

    Step 1. Draw the cylinder. Label the radius r and the height h.

    Step 2. Express the volume V as a function of r and h. Express the surface area of the cylinder as a function of r and h. Equate the surface area to 100.
    Think of these as V being a function of r, and h also being a function of r.

    Step 3. (a) Differentiate *implicitly* the volume function and the surface area function with respect to r.
    (b) Use the surface area function after it has been differentiated to get an expression for $D_r h$. Substitute $D_r h$ into the $D_r V$ expression.

    Step 4. Let $D_r V = 0$. Find a critical relationship between r and h, or a critical value for r.
    Substitute into the surface area equation.
    Find the largest possible volume. Check to make sure it is truly a maximum for the entire domain of r.

61. (a) Find the dimensions of the right circular cylinder of maximum volume which can be inscribed in a right circular cone of altitude 10 and radius 12.
    (b) What is the result for a cone of altitude H and radius R?

    Step 1. Draw the diagram of a cone with a cylinder inside it. The diagram should be a cross-section viewed from the side. Label the altitude and radius of the cone with fixed numerical values. Label the cylinder with height, h, and radius, r. These quantities are varying.

    Step 2. Write the volume of the cylinder (call it V) as a function of h and r.
    Using geometry (i.e. look for similar triangles), write down a relationship between r and h. You now have two relations involving two variables, r and h. Think of either r or h as your independent variable.

    Step 3. Differentiate *both* relations *implicitly* with respect to your chosen variable.
    The second relation will have a $D_h r$ or $D_r h$ in it, which you may solve for in terms of r and h. Substitute your expression for $D_h r$ or $D_r h$ back into V.
    At this stage you should have V' as a function of r and h.

    Step 4. Let $V' = 0$ and get a critical relation between r and h. Finish the question that was originally asked.

62. Find the dimensions of a right circular cylinder of maximum volume V which has a given surface area S.

    Step 1. Draw a cylinder labelling the height h and the radius r.

    Step 2. Express V as a function of both h and r. Express S as a function of both h and r. Decide the letter which is to be the independent variable (for purposes of differentiation). Before going on consider each letter. Which is fixed?

    Step 3. Differentiate both expressions implicitly with respect to the independent variable. By an appropriate substitution get V in terms of r and h. Finish it.

63. Find the dimensions of a right circular cylinder of minimum surface area S which has a given volume V.

    Step 1. Exactly the same as in # 62.

    Step 2. Exactly the same as in # 62.

    Step 3. Differentiate both expressions implicitly with respect to the independent variable. By an appropriate substitution get S' in terms of r and h. Finish it.

64. Find the dimensions of the rectangle of maximum area which can be inscribed in the ellipse $(x^2/16) + (y^2/9) = 1$.

    Step 1. Draw the ellipse around the origin. (If you have forgotten, the major axis is from -4 to $+4$, and minor axis is from -3 to $+3$). Draw the rectangle inside the ellipse oriented so that the sides are parallel to the axis. Let the point in the first quadrant where the rectangle touches the ellipse be (x, y).

    Step 2. Write down the area, A, of the rectangle in terms of x, and y. Write down the equation of the ellipse. Think of either x, or y, as the independent variable.

    Step 3. Differentiate both relations implicitly with respect to the selected independent variable. Find A' in terms of x and y by appropriate substitution. Finish it.

65. Find the dimensions of the rectangle of maximum perimeter which can be inscribed in the ellipse $(x^2/a^2) + (y^2/b^2) = 1$.

    Step 1. Exactly the same as in # 64.

    Step 2. Write down the perimeter, P, of the rectangle in terms of x and y. Write down the equation of the ellipse. Think of either x or y as the independent variable.

    Step 3. Differentiate both relations implicitly with respect to the independent variable. Find P' in terms of x and y by a suitable substitution. Finish it.

66. The stiffness of a given length of beam is proportional to the product of the width and the cube of the depth. Find the shape of the stiffest beam which can be cut from a cylindrical log (of the given length) with cross-sectional diameter of 4 ft.

    Step 1. Draw a circle around the origin representing the cross-section of the log. Inside of the circle, draw the cross-section of the beam, labelling the width w and the depth d. The corners of the rectangle should just touch the boundary of the circle. Label the corner in the first quadrant with the appropriate co-ordinates as expressions in w and d.

    Step 2. Write the stiffness, S, as a function of w and d. Write an equation for a circle of diameter 4 ft. Rewrite the equation of the circle with w and d in it. Select either w or d as the independent variable.

    Step 3. Differentiate both relations implicitly and finish it.

67. (a) A manufacturer makes aluminum cups of a given volume (16 in^3) in the form of right circular cylinders
open at the top. Find the dimensions which use the least material.

(b) What is the result for a given volume V?

Step 1. Draw the diagram. Label the radius of the cup r and the height h.

Step 2. Write the volume V as a function of r and h. V is a constant (16 in^3). Write the surface area S as a function of r and h. What quantity has to be minimized? Think of a variable which will be considered independent.

Step 3. Differentiate *both* functions *implicitly* and finish it.

68. One number plus the square of another number totals 50. Select the numbers so that their product is as large as possible.

Step 1. On this *rare* occasion you may skip the diagram. Call one number x, and the other y.

Step 2. Translate the first sentence of the question into a mathematical expression involving x and y. Write the product P in terms of x and y. Think of either x or y as the independent variable.

Step 3. Differentiate *both* mathematical expressions implicitly with respect to your independent variable. Finish it.

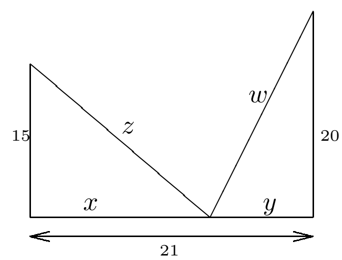
69. The product of two numbers is 16. Determine them so that the square of one plus the cube of the other is as small as possible.

Step 1. Call one number x and the other y.

Step 2. Use each sentence in the question given to find the two mathematical expressions in terms of x and y. What are you trying to minimize? (Give the quantity to be minimized a letter name, say Q if you have not already done so.) Pick on either x or y as your independent variable for the rest of the problem. Finish it.

70. Two vertical poles 15 and 20 ft. high are spaced 21 ft. apart. The top of each pole is to be joined by a guy wire to a stake in the ground; the stake is located on a direct line between the poles. Where should the stake be placed in order to use the least amount of wire?

Step 1. Drawing is at the right with all the possible variables. Let L be the total length of the wire.



Step 2. Express L in terms of z and w. Write down other relationships between the variables. You need four distinct relations because there are four variables. Think of one variable as the independent one, say x.

Step 3. You may differentiate all 4 relations *w.r.t.* x. Then by substitutions, arrange to get L' in terms of x, y, w, z. Then get L' in terms of x only by using substitutions from the relations found in Step 2. Finish it.

71. Find the dimensions of the cylinder of greatest lateral area which can be inscribed in a sphere of given radius R.

   Step 1. Draw the cross-section of a cylinder inside a sphere. The diagram should look like a rectangle inside a circle. Label the radius of the circle R, the radius of the cylinder x, and the height of the cylinder as $2y$. Which quantity is a fixed value, and which are variables?

   Step 2. What is the volume V of the cylinder in terms of x and y? Find another relation involving x and y from the diagram. Decide on a variable which will be considered the independent one. Finish it.

72. A piece of wire of length L is cut into two parts, one of which is bent into the shape of a square and the other into the shape of a circle. (a) How should the wire be cut so that the sum of the enclosed areas is a minimum? (b) How should it be cut to get the maximum enclosed areas?

   Step 1. Draw circle and the square. Label the radius of the circle r and the length of the side of the square x.

   Step 2. What quantity is fixed in the problem? What quantity do you want to minimize or maximize? Write a function for the sum of the perimeter of the square and the circumference of the circle. Write a function for the sum of the area of the square and the area of the circle. Pick on either r or x as your independent variable.

   Step 3. Differentiate implicitly and finish it.

73. A piece of wire of length L is cut into two parts, one of which is bent into the shape of an equilateral triangle and the other into the shape of a circle. How should the wire be cut so that the sum of the enclosed areas is (a) a minimum? (b) a maximum?

   Step 1. Draw the circle and the triangle. Label the radius of the circle r and the base of the equilateral triangle x.

   Step 2. What quantity is fixed in the problem? What quantity do you want to minimize or maximize? Write a function for the sum of the perimeter of the triangle and the circumference of the circle. Write a function for the sum of the area of the triangle and the area of the circle. Finish it.

74. Find the dimensions of the right circular cone of maximum volume which can be inscribed in a sphere of given radius R.

   Step 1. Draw a cross-section of a cone inside a sphere. The diagram should look like a triangle inside a circle. Label the radius of the circle R, the radius of the cone x and the height of the cone h. Which quantity is fixed and which are variable?

   Step 2. Express the volume V of the cone as a function of x and h. Join the centre of the sphere to the circular rim of the cone. Write a relationship involving x, h, and R. Pick on one variable, say x, to be thought of as the independent one. Finish it.

75. Find the dimensions of the right circular cone of minimum volume which can be circumscribed about a sphere of radius 12. Show that this minimum volume is twice that of the sphere.

   Step 1. Draw a cross-section of a sphere inside a cone. The diagram should look like a circle inside of a triangle. Label the radius of the cone r and the height of the cone h. These are the two variables. Draw a line from the centre of the sphere to the place where the sphere touches the slant height of the cone. Label the radius of the sphere, 12.

   Step 2. Express the volume of the cone as a function of r and h. Mark right angles in the diagram. Look for a pair of similar triangles. Using ratios of the sides write down a relationship involving r and h. Finish it.

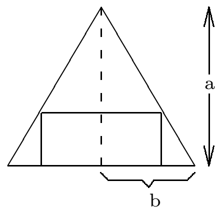
76. What are the proportions of the cone of given volume that has the minimum total surface area (bottom plus lateral surface)?

    Step 1. Draw a cone. Label the radius of the bottom r, and the height of the cone h.

    Step 2. What quantity is given as fixed? What quantity is to be minimized? Write the volume V of the cone as a function of r and h. Write the surface area of a cone including the area of the bottom surface. Finish it.

77. Find the dimensions of the rectangle of maximum area that can be inscribed as shown in an isosceles triangle.

    Step 1. Copy the diagram given. Label the height of the rectangle as y and the length of the rectangle as $2x$. What distances are fixed values, and what distances are variables?



    Step 2. What is the area A of the rectangle in terms of x and y? Using the *geometry* of the diagram find another relationship involving x and y. Select one of the variables as the independent one. Finish it.

78. The product of two positive numbers is a given number A. How is the first related to the second if the sum of the first plus twice the second number is a minimum?

    Step 1. Let the first number be x and the second number be y.

    Step 2. What quantity is fixed? What quantity is to be minimized? Write the product, A, as a function of x and y.

    Write the mathematical expression for the situation described in the second sentence. Pick either x or y as the independent variable. Finish it.

79. What circular sector with a given perimeter has greatest area?

    Step 1. Draw the diagram of a circular sector. (A sector is part of a circle like a piece of pie.) Label the angle of the sector, θ, and the radius, r. These are variables.

    Step 2. What quantity is fixed? What quantity do we wish to minimize? What is the perimeter of the sector as a function of r and θ? What is the area of the sector as function of r and θ? Think of either r or θ as the independent variable. Finish it.

80. The stiffness of a beam of rectangular cross section is jointly proportional to its width and the cube of its depth. Find the stiffest beam that can be cut from a log of diameter a.

    Step 1. Draw the diagram of a cross-section of a log showing where the rectangular beam will be cut. The drawing should look like a circle with a rectangle in it. Label the width of the rectangle w and the depth d. Label the diameter a. Is it fixed in value?

    Step 2. Use the first sentence of the problem to write the stiffness, S, as a function of w and d. Using the geometry of the situation write another relationship involving w and d. Finish it.

81. A gas tank of volume V is to be made in the shape of a cylinder (with a flat bottom to it), surmounted by a hemisphere. What should be the proportions for minimum material?

    Step 1. Draw a cylinder with a hemispherical top. Label the radius of the cylinder r and the radius of the top is also r. Label the height of the cylinder (not counting the hemisphere on top) as h.

Step 2. What quantity is fixed in value? What quantity is to be minimized. Find the volume V as a function of r and h. Think of either r or h as the independent variable. Finish it.

82. A tank is to be made as in Problem 81 but with a fixed total surface area. What should its proportion be for maximum volume?

83. Find the dimensions of the rectangle of maximum area that can be inscribed in the ellipse $b^2x^2+a^2y^2 = a^2b^2$.

84. Find the right triangle of greatest area that has a hypotenuse of given length C.

85. Find the isosceles triangle of least area that can be circumscribed about a circle of radius a.

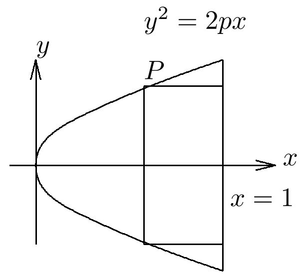
86. Find the isosceles triangle of greatest area that can be inscribed in a circle of radius a.

87. A water tank is to have a square base and open top and contain 1000 gal. If the base is twice as costly as the sides, what proportions give minimum cost?

88. A page of a book is to contain 24 sq. in. of print. If margins at the top and bottom of the page are $1\frac{1}{2}$ in. and at the sides 1 in., what is the size of the page of least area?

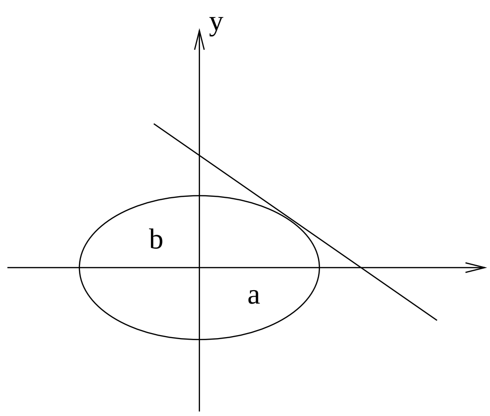
89. At what point does the tangent line to $y = \sqrt{x+1}$ make, with the axes, a right triangle of least area?

90. At what point P does the rectangle with a vertex at P have maximum area?



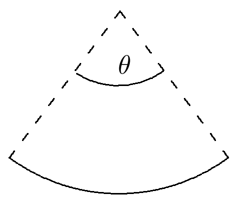
91. A Norman window has the shape of a rectangle surmounted by a semicircle. For a given perimeter what proportions give greatest area?

92. What is the minimum area for the triangle formed by the axes and the tangent line to the ellipse with semi-axes a and b?



93. For the figure of Problem 92, what is the least length cut out from the tangent by the axes?

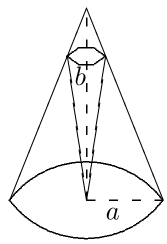
94. A strip of sheet metal of width c is to be bent to form a circular trough. What should the angle θ be for maximum carrying capacity?



95. Find the cone of maximum lateral surface area that can be inscribed in a sphere of radius a.

96. Find the cone of maximum total surface area that can be inscribed in a sphere of radius a.

97. In a cone of height b and radius of base a another cone is inscribed upside down. Find the dimensions of the inscribed cone of maximum volume.



98. A fence $13\frac{1}{2}$ ft. high is 4 ft. from the side wall of a house. What is the length of the shortest ladder, one end of which will rest on the level ground outside the fence and the other on the side wall of the house?

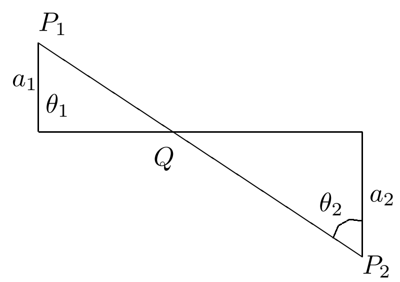
99. A silo is to be built in the form of a right circular cylinder surmounted by a hemisphere. If the cost of the material per square foot is the same for floor, walls, and top, find the most economical proportions for a given capacity V.

100. Work Problem 99, given that the floor costs twice as much per square foot as the sides and the hemispherical top costs three times as much per square foot as the sides.

101. A tank is to have a given volume V and is to be made in the form of a right circular cylinder with hemispheres attached at each end. The material for the ends costs twice as much per square foot as that for the sides. Find the most economical proportions.

102. Find the length of the longest rod which can be carried horizontally around a corner from a corridor 8 ft. wide into one 4 ft. wide. (*Hint:* Observe that this length is the minimum value of certain lengths.)

103. Diagram of a ray of light entering the water.

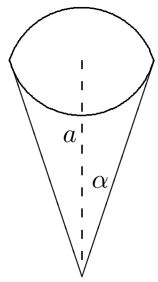


Suppose the velocity of light is V_1 in air and V_2 in water. A ray of light travelling from a point P_1 above the surface of the liquid to a point P_2 below the surface will travel by the path which requires the least time. Show that the ray will cross the surface at the point Q in the vertical plane through P_1 and P_2 so placed that

$$\frac{\sin\theta_1}{V_1} = \frac{\sin\theta_2}{V_2}$$

where θ_1 and θ_2 are the angles shown in the diagram.

104. Into a full conical wineglass of depth a and generating angle α there is carefully dropped a sphere of such size as to cause the greatest overflow. Show that the radius of the sphere is $\frac{a\sin\alpha}{(\sin\alpha+\cos 2\alpha)}$.



105. The intensity of illumination at any point is proportional to the product of the strength of the light source and the inverse of the square of the distance from the source. If two sources of relative strengths a and b are a distance apart, at what point on the line joining them will the intensity be a minimum? Assume the intensity at any point is the sum of intensities from the two sources.

106. If the sum of the areas of a cube and sphere is constant, what is the ratio of an edge of the cube to the diameter of the sphere when
   (a) the sum of their volumes is a minimum,
   (b) the sum of their volumes is a maximum?

107. Two towns, located on the same side of a straight river, agree to construct a pumping station and filtering plant at the river's edge, to be used jointly to supply the towns with water. If the distances of the two towns from the river are a and b and the distance between them is c, show that the sum of the lengths of the pipe lines joining them to the pumping stations is at least as great as $\sqrt{c^2+4ab}$.

108. Light emanating from a source A is reflected to a point B by a plane mirror. If the time required for the light to travel from A to the mirror and then to B is a minimum, show that the angle of incidence is equal to the angle of reflection.

109. Two paths intersect at right angles in dense woods. Another straight path is to cut through the woods forming a triangle. This path is intended to pass by a spring which is 27 ft. from one road and 64 ft. from the other road (measured perpendicularly from the spring to the road). Find where to cut the path that is the shortest.

110. For the situation described in Problem 109, find the path that cuts off the least area in the triangle.

111. Find the shortest distance from the point $(0, 2)$ to the hyperbola $x^2 - y^2 = \theta$.

112. It is required to make a tin container to hold 27 cu. in., in the form of a right circular cylinder. If the top and bottom of the can are cut from the square sheets, and the corner pieces are wasted, find the radius of the container which requires the least tin.

113. Show that a quart tomato can has least surface area if its height is equal to the diameter of its base.

114. Show that of all rectangles having a given area, the square has the least perimeter; and of all rectangles having a given perimeter the square has the largest area.

115. What is the most economical shape for a floorless, conical tent if the volume to be enclosed by the tent is specified?

116. A lighthouse is at point P, 3 miles offshore, from the nearest point 0 of a straight beach. A store is located 5 miles down the beach from 0. The lighthouse keeper can row 3 miles/hr and walk 4 miles/hr. How far along the beach from 0 should he land in order to get to the store in the shortest possible time?