Contents

Chapter 9

Newton's Method

9.1 Solving Equations with Error Analysis Using Newton's Method

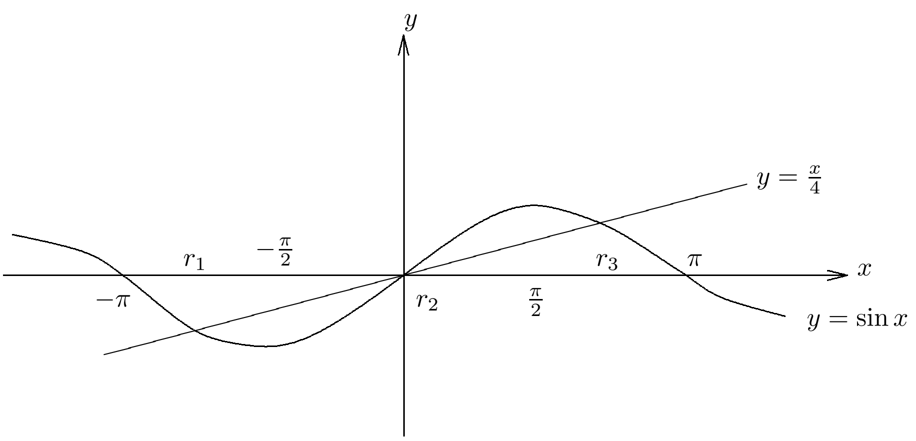
Read the Gold Notes, V.A Newton's Method in the Special Case.

Definition of accuracy to K decimal places: If E is the error in an approximation then the approximation will be considered accurate to K decimal places if $|E| < 0.5 \times 10^{-K}$.

Example: Solve $\sin x - \frac{x}{4} = 0$ with an error less than $\pm 1 \times 10^{-3}$.

Step 0: Find the number of roots and suitable intervals each containing exactly one root. One method is to write $f(x) = \sin x - \frac{x}{4}$ and use graph sketching techniques to sketch $f(x)$. (You are not allowed to plot points.) Since $f(x)$ is a differential function on $\mathbb{R}$, it is continuous on $\mathbb{R}$. Hence the intermediate value theorem applies. Use this theorem to test and reduce the size of each interval that your graph shows containing a root. Also make sure $f'(x) \neq 0$ in each of the above intervals.

For this problem another method works. Write $\sin x = \frac{x}{4}$ and graph $\sin x$ and $\frac{x}{4}$ on the same coordinate system. The roots of $\sin x - \frac{x}{4} = 0$ occur where the two graphs intersect. If the graphs do not intersect the equation has no roots (an example is $\ln x - x = 0$). It is important to know that $\sin x < x$ if $x > 0$. This result implies $\sin x - x = 0$ has only one root.



The sketch shows there are exactly three roots with the following properties

(a) $r_2 = 0$ (by inspection)

(b) $r_3 \in \left(\frac{\pi}{2}, \pi\right)$

(c) $r_1 \in \left(-\pi, -\frac{\pi}{2}\right)$

(d) $r_1 = -r_3$ (by symmetry; i.e., both functions are odd).

We need only obtain r_3 to determine all three roots. To find a suitable interval compute

$$f(2) \approx 0.409 \qquad \text{and} \qquad f(3) \approx -0.608$$

$f(2)$ and $f(3)$ have opposite signs implies $r_3 \in [2,3]$ by the intermediate value theorem. Next we show $f'(x) \neq 0$ in $[2,3]$.

$f''(x) = -\sin x \Rightarrow f'(x)$ is decreasing in $[2,3]$. Also $f'(2) \approx -0.666$.

$f'(2) < 0$ and $f'(x)$ decreasing in $[2,3]$ implies $f'(x) < 0$ in $[2,3]$; i.e., $f'(x) \neq 0$ in $[2,3]$.

Step 1: Determine constants M_1 and M_2 so that

$$|f'(x)| = \left|\cos x - \frac{1}{4}\right| > M_1 > 0 \qquad x \in [2,3]$$

and

$$|f''(x)| = |-\sin x| < M_2 \qquad x \in [2,3].$$

In Step 0 we saw that $f'(x) < 0$ in $[2,3]$. This means

$\left|\cos x - \frac{1}{4}\right| = \frac{1}{4} - \cos x$. $\left(\frac{1}{4} - \cos x\right)' = \sin x > 0$ on $[2,3]$. Therefore

$\left|\cos x - \frac{1}{4}\right|$ is an increasing function on $[2,3]$ implies the minimum value of

$|f'(x)| = |f'(2)| \approx .66614683$. Choose $M_1 = .6$.

$|-\sin x| = |\sin x| = \sin x$ on $[2,3]$. Since $\sin x$ is a decreasing function on $[2,3]$, the maximum value is

$\sin 2 \approx .9093$. Choose $M_2 = .91$.

$K = \frac{M_2}{2M_1} = \frac{.91}{1.2} \approx .75833333$.

Find $h > 0$ such that $Kh \leq 1$.

$h \leq \frac{1}{.7583} \approx 1.319$. Pick $h = 1.1$.

Step 2: Find inside $[2,3]$ an interval $[u,v]$ which contains r_3 and such that $v - u < h$. (See Gold Notes.) Since the length of $[2,3] = 1$ and $h = 1.1$, the interval $[2,3]$ does the job.

Step 3: Compute the minimum number of iterations need for the $|\text{error}| < \frac{1}{1000}$ (see step 6, Gold Notes). The absolute error in the ith iteration is

$$|r - x_1| \leq \frac{1}{K}\left(\frac{Kh}{2}\right)^{2^i}$$

$$\frac{1}{K} \approx 1.318681319 < \frac{4}{3}$$

$$\begin{aligned} Kh &\approx .83416666 < 1 \\ |r - x_1| &\leq \frac{4}{3}\left(\frac{1}{2}\right)^2 = \frac{1}{3} \\ |r - x_2| &\leq \frac{4}{3}\left(\frac{1}{2}\right)^4 = \left(\frac{1}{3}\right)\left(\frac{1}{4}\right) \\ |r - x_3| &\leq \frac{4}{3}\left(\frac{1}{2}\right)^8 = \left(\frac{1}{3}\right)\left(\frac{1}{2^6}\right) \\ |r - x_4| &\leq \frac{4}{3}\left(\frac{1}{2}\right)^{16} = \frac{1}{3}\left(\frac{1}{2^{14}}\right) \end{aligned}$$

But $\frac{1}{2^{10}} = \frac{1}{1024} \Rightarrow \frac{1}{3}\left(\frac{1}{2^{14}}\right) < \frac{1}{1000}$. Only 4 iterations are needed.

Step 4: (See Step 3 Gold Notes).

$$\begin{aligned} x_0 &= \frac{2+3}{2} = \frac{5}{2} \\ x_1 &= x_0 - \frac{\sin x_0 - \frac{x_0}{4}}{\cos x_0 - \frac{1}{4}} = -\frac{4\sin x_0 - x_0}{4\cos x_0 - 1} + x_0 \end{aligned}$$

Calculator program.

Store x_0 in memory.

A: (RM sin ×4− RM) ÷

(RM cos ×4 − 1) + /−

+ RM =

Record answer and store in memory then go to A and repeat 3 more times.

$$\begin{aligned} x_1 &= 2.474762863 \\ x_2 &= 2.474576798 \\ x_3 &= 2.474576787 \\ x_4 &= 2.474576787. \end{aligned}$$

Example: Solve the equation $e^x - x^2 = 0$ within $\pm .0005$.

Step 0: Write $f(x) = e^x - x^2$. The roots of f occur where $e^x = x^2$. Graph e^x and x^2 on the same coordinate system and observe they intersect at a negative value of x. This indicates there is only one root r and $r < 0$.

f is a continuous function on $\mathbb{R}$ because its derivative exists on $\mathbb{R}$. Therefore we can apply the intermediate value theorem to obtain an interval, I, containing r:

$$f(-1) \approx -.632 \qquad \text{and} \qquad f(-\frac{1}{2}) \approx 3.57.$$

The intermediate value theorem guarantees at least one root in the interval $I = [-1, -\frac{1}{2}]$ and $f'(x) \neq 0$ on I because $f'(x) = 0 \Leftrightarrow e^x - 2x = 0 \Leftrightarrow e^x = 2x$. Since $e^x > 0$ and $2x < 0$ for $x \in I$, $f'(x) \neq 0$ on I.

Step 1: (Find M_1 and M_2 and choose h.)

$|f'(x)| = |e^x - 2x| = e^x - 2x$ because both e^x and $-2x$ are positive on I $(e^x - 2x)' = e^x - 2 < 0$ for $x < 0$. This means that $|f'(x)|$ is decreasing on I. Therefore the minimum value of $|f'(x)|$ on I is $|e^{-\frac{1}{2}} + 1| \approx 1.606$. Pick $M_1 = 1.6$

$|f''(x)| = |e^x - 2|$. Since $e^x < 1$ for $x < 0$, $e^x - 2 < 0$ on I implies $|e^x - 2| = 2 - e^x$.

$(2-e^x)' = -e^x < 0$ implies $|e^x - 2|$ is decreasing on I. The maximum value of $|f''(x)|$ on I is $|e^{-1} - 2| \approx 1.632$. Pick $M_2 = 1.64$.

$$K = \frac{M_2}{2M_1} = \frac{1.64}{2(1.6)} \approx 0.5125 < \frac{6}{10} = \frac{3}{5}.$$

For $Kh \leq 1$, $h \leq \frac{1}{0.5125} \approx 1.95$. Pick $h = 1$.

Step 2: Since the length of $I = \frac{1}{2} \leq h$ we choose $[u, v] = [-1, -\frac{1}{2}]$ and $x_0 = -\frac{3}{4}$.

Step 3: Use the fact that $|r - x_0| < \frac{h}{2}$ and $|r - x_{i+1}| \leq K|r - x_i|^2$ to compute the minimum number of iterations required for the absolute value of the error to be less than .0005.

$$\begin{aligned}
|r - x_0| &< \frac{1}{4} \\
|r - x_1| &< \frac{3}{4}\left(\frac{1}{4}\right)^2 = .046875 < .05 \\
|r - x_2| &< \frac{3}{4}(.05)^2 = .001875 < .002 \\
|r - x_3| &< \frac{3}{4}(.002)^2 = 3 \times 10^{-6} < .00005.
\end{aligned}$$

Step 4:

$$\begin{aligned}
x_0 &= -.75 \\
x_1 &= -.7043018781 \\
x_2 &= -.7034676979 \\
x_3 &= -.7034674225.
\end{aligned}$$

Within an accuracy of $\pm.00005$, the root is $-.70347$.

Example: Solve $f(x) = 0$, $x \in [-2, 2]$ with accuracy of $\frac{5}{10^5}$ where $f(x) = x^3 + x^2 + 1$.

A. Find all subintervals of $[-2, 2]$ that contain exactly one root of $f(x) = x^3 + x^2 + 1$

$$f'(x) = 3x^2 + 2x = x(3x + 2) = 0 \Leftrightarrow x = 0 \quad \text{or} \quad -\frac{2}{3}$$

$$\begin{aligned}
f(-2) &= -3 \\
f(-2/3) &= \tfrac{31}{27}
\end{aligned}
\left\{ \begin{array}{l} \text{Using } IVT \text{ and Rolles Theorem} \\ \text{we can show that } [-2, -\tfrac{2}{3}] \\ \text{contains exactly one root } r \end{array} \right\}$$

$$\begin{aligned}
f(0) &= 1 \\
f(2) &= 13
\end{aligned}
\left\{ \begin{array}{l} \text{No change of sign here} \Rightarrow \text{no} \\ \text{more roots in } [-2, 2] \end{array} \right\}$$

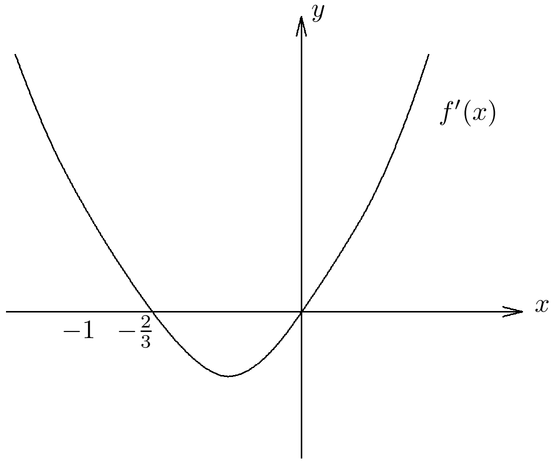
Now $f(-2) = -3$ and $f(-1) = 1 \Rightarrow r \in [-2, -1]$ with the requirement that $f'(x) \neq 0$ in $[-2, -1]$.

B. Apply Newton's method to the interval $[-2,-1]$. We can pick a smaller interval containing r now to save work later:

$$\begin{aligned} f(-3/2) &= -\frac{27}{8}+\frac{9}{4}+1 \\ &= -\frac{27}{8}+\frac{18}{8}+\frac{8}{8} \\ &= -\frac{1}{8} \quad \text{and} \quad f(-1)=1 \\ &\Rightarrow r\in[-3/2,-1] \end{aligned}$$

write $I=[-3/2,-1]$.

C. Find M_1 and M_2 $(M_1 \le |f'(x)|$ on I and $|f''(x)| \le M_2$ on I.) $f'(x)$ is a quadratic that is positive and decreasing if $x<-\frac{2}{3}$. So $|f'(x)| = 3x^2+2x$ on I and since $|f'(x)|$ is decreasing on I its minimum value is $|f'(-1)|=1$. Choose $M_1=1$.



Another way to find M_1 is to observe and state that $f''(x)=6x+2$, the first derivative of $f'(x)$, is non zero on I so minimum cannot occur at a critical point so it must occur at an endpoint.

Now we find M_2, $f''(x)=6x+2$. $f''(x)<0$ if $x<-\frac{1}{3}$. So on $I=[-3/2,-1]$

$|f''(x)| = -6x-2$ (definition of absolute value)

$|f''(x)|$ is decreasing (negative slope) $\Rightarrow$ maximum occurs at $-\frac{3}{2}$ with value 7 (another way: $|f''(x)| = |6x+2| \le 6|x|+2 < 6|\frac{3}{2}|+2=11$).

$|f''(x)| \le M_2 = 7$ $K=\frac{M_2}{2M_1}=\frac{7}{2}$. Pick $h=\frac{2}{7}$ then $Kh=1$.

D. Find $[u,v]$ containing r such that length of $[u,v] \le h = \frac{2}{7}$. Note $\frac{2}{7}=\frac{8}{28}>\frac{7}{28}=\frac{1}{4}$. Partition I in half:

$-\frac{3}{2}$	$\frac{1}{4}$	$-\frac{5}{4}$	$\frac{1}{4}$	-1

$$\begin{aligned} f\left(-\frac{5}{4}\right) &= \frac{-125}{64}+\frac{25}{16}+1=\frac{-125+100+64}{64}>0 \\ f\left(-\frac{3}{2}\right) &= -\frac{1}{8}<0 \quad \text{Hence } r\in[-\tfrac{3}{2},-\tfrac{5}{4}] \\ x_0 &= \frac{1}{2}\left(-\frac{6}{4}-\frac{5}{4}\right)=\frac{-11}{8} \end{aligned}$$

E. Determine the minimum number of iterations for the Newton formula required for the given accuracy of $\frac{1}{10^5}$

$$|r - x_i| \leq \frac{1}{K}\left(\frac{Kh}{2}\right)^{2^i} < \frac{1}{10^5}$$

$$|r - x_i| \leq \frac{2}{7}\left(\frac{1}{2}\right)^{2^i} < \frac{1}{(2)^{2^i}} < \frac{1}{10^5}$$

$$i = 3 \quad : \quad \frac{1}{2^8} = \frac{1}{256}$$

$$i = 4 \quad : \quad \frac{1}{2^{16}} = \frac{1}{(256)^2} = \frac{1}{65536}$$

F. Newton Iteration formula

$$w_1 = x_1 = x_0 - \frac{x_0^3 + x_0^2 + 1}{3x_0^2 + 2x_0} \qquad x_0 = \frac{-11}{8}$$

$$x_1 = x_0 - \frac{\left((x_0 + 1)x_0^2 + 1\right)}{(3x_0^2 + 2x_0)}$$

$$x_1 = -1.47459893$$

$$x_2 = -1.465649097$$

$$x_3 = -1.465571238$$

$$x_4 = -1.465571232 \qquad \text{stop here for required accuracy}$$

$$x_5 = -1.465571232$$

NOTES: In our error analysis for Newton's method we need to compute bounds on functions with respect to a given interval. Since the upper or lower bound may *NOT* occur at the endpoints of the interval you are required to justify your bound is correct. Read IV.C in the Gold Notes and Chapter 7 of these Redbook Notes.

PROBLEMS

1. Find the positive root of $x^2 = 5$ to within $\pm.002$.
2. Find the smallest root of $x^2 - 100x + 2499.96 = 0$ to within $\pm.0001$.
3. Find the root of $x^3 + 2x + 4 = 0$ to within $\pm.0001$.
4. Find all roots of $x^\pi + x = 1$ within $\pm.00005$.
5. Find the positive root of $x^4 - x^2 + x - 1 = 0$ to within $\pm.00005$.
6. Find the positive root of $\sin x - \frac{x}{2} = 0$ to within $\pm.0001$.
7. Find the first positive root of $\tan x - x = 0$ to within $\pm.002$.
8. Find the second smallest positive root of $\tan x - x = 0$ to within $\pm.002$.
9. Find the real root of $10x^3 + 10x + 1 = 0$ to within $\pm.001$.
10. Find the real root of $x^3 - 10^{-6}x - 1 = 0$ to within $\pm 10^{-10}$.

11. Find the roots of $2x = 3\sin x$ within $\pm 10^{-6}$.
12. Using $x_0 = -\frac{3}{4}$ as an estimate for a root of $x^3 + x + 1$, find x_1, x_2, x_3 by three iterations of Newton's Method. Give an upper bound on the error.
13. What is the integer in the 30th decimal place of the positive root of $x^2 + 2x - 10^{-29}$.
14. Find a root of $2xe^x = 1$ to within $\pm .0005$.
15. Find the negative root of $e^{-x} + x - 1 = 0$ to within $\pm .0005$.
16. Find the negative root of $e^x - x - 1 = 0$ to within $\pm .0005$.
17. Find the root of $e^x - x^2 = 2$ to within $\pm .0005$.
18. Find the first positive root of $\tan x - 2x = 0$ to within $\pm .002$.
19. $x^2 + 4\sin x = 0$. Find all roots within $\pm .0001$.
20. $\cos x - 10x = 0$. Find all roots within $\pm .0002$.
21. $-2x = e^x$. Find the negative root within $\pm .0001$.
22. $e^x = 2 - x$. Find all the roots within $\pm .0001$.
23. $\ln x - \sqrt{x} = 0$. Find all roots within $\pm .0001$.
24. $e^x + x = 0$. Find all roots within $\pm 10^{-6}$.
25. $\cos x - x^2 = 0$. Find all roots within $\pm 10^{-6}$.
26. $\ln x - \frac{1}{x} = 0$. Find all roots within $\pm 10^{-6}$.
27. $e^x - 2 - x = 0$. Find all roots within $\pm .0001$.
28. $2x - x^2 + \ln x = 0$. Find all roots within $\pm 10^{-6}$.
29. $\ln x + \cos x = 0$. Find all roots within $\pm 10^{-6}$.
30. $x^2 - 4 + \ln x = 0$. Find all roots within $\pm 10^{-6}$.
31. Let $f(x) = x - 2 + \ln x$. How may roots does $f(x)$ have? (Justify your answer.) Use Newton's method to find all real roots of $f(x)$ with an error of at most 10^{-6}.
32. Follow the instructions in question 31 for the function: $f(x) = e^x + \frac{3x}{2}$.